Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 146 (2007) 115–148
Journal of Approximation Theory
www.elsevier.com/locate/jat

Completeness of an exponential system in weighted Banach spaces and closure of its linear span

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Received 26 March 2006; received in revised form 5 November 2006; accepted 1 December 2006

Communicated by Tamás Erdélyi
Available online 31 December 2006

Abstract

For a real multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$, that is, a sequence where $\{\lambda_n\}$ are distinct positive real numbers satisfying $0 < \lambda_n < \lambda_{n+1} \mapsto \infty$ as $n \mapsto \infty$ and where each λ_n appears μ_n times, we associate the exponential system

$$E_\Lambda = \{t^k e^{\lambda_n t} : k = 0, 1, 2, \ldots, \mu_n - 1\}_{n=1}^{\infty}.$$

For a certain class of multiplicity sequences, we give necessary and sufficient conditions in order for E_Λ to be complete in some weighted Banach space of continuous functions on $\mathbf{R}$, and in some weighted $L^p(-\infty, \infty)$ spaces of measurable functions, with $p \in [1, \infty)$. We also prove that if E_Λ is incomplete in the weighted spaces, then every function in the closure of the linear span of E_{Λ^*}, where $E_{\Lambda^*} = \{t^{\mu_n - 1} e^{\lambda_n t}\}_{n=1}^{\infty}$, can be extended to an entire function represented by a Taylor–Dirichlet series

$$g(z) = \sum_{n=1}^{\infty} c_n z^{\mu_n - 1} e^{\lambda_n z}, \quad c_n \in \mathbf{C}.$$

Furthermore, we prove that E_Λ is minimal in the weighted spaces if and only if it is incomplete.

MSC: 30B60; 30B50

Keywords: Completeness; Closure; Minimality; Taylor–Dirichlet series

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2006.12.002

1. Introduction

Malliavin in [15] considered the following problem in the sense of Bernstein's weighted polynomial approximation on the real line [9,8,16,14,6,5]. Let $W(t)$ be a real-valued continuous function defined on the interval $[0, +\infty)$. Assume that $\log|W(e^s)|$ is a convex function of s. Denote by C_W the set of complex-valued continuous functions defined on the interval $[0, +\infty)$, such that a function $f \in C_W$ if $\lim_{t\to+\infty} f(t)/W(t) = 0$. Let $\|f\|_W = \sup\{|f(t)/W(t)| : t \in [0, +\infty)\}$, thus C_W equipped with this norm becomes a weighted Banach space. Let also $\{\lambda_n\}$ be an increasing sequence of positive numbers tending to infinity so that the $\liminf_{n\to\infty}(\lambda_{n+1} - \lambda_n) > 0$. Malliavin [15, Theorem 8.3] proved that the system $\{t^{\lambda_n}\}_{n=0}^{\infty}$ is incomplete in C_W if and only if there exists $\eta \in \mathbf{R}$ such that

$$\int_1^{+\infty} \frac{\log|W(e^{\sigma_\Lambda(t)-\eta})|}{t^2}\,dt < \infty,$$

where $\sigma_\Lambda(t) = \sum_{\lambda_n \leqslant t} 1/\lambda_n$. When $W(t) = e^t$ the result is due to Fuchs [12]. We also note that if the system $\{t^{\lambda_n}\}$ is incomplete in C_W and λ_n are positive integers, the question of the closure of its linear span was considered by Anderson and Binmore [1]. They proved that if a function $f \in \overline{span}\{t^{\lambda_n}\}$ in the $\|\ \|_W$ norm, then f is extended to an entire function with a gap power series expansion of the form $f(z) = \sum a_n z^{\lambda_n}$.

Motivated by the methods of Malliavin, Deng [11] obtained a result regarding the completeness of the *exponential* system $\{e^{\lambda_n t}\}$ in some weighted Banach space C_α of continuous functions, where the frequencies λ_n are complex numbers with positive real part, that lie in an angle of opening less than π, and satisfy the spacing condition $\liminf_{n\to\infty}(|\lambda_{n+1}| - |\lambda_n|) > 0$. The space C_α consists of all the complex-valued continuous functions on $\mathbf{R}$, so that the $\lim_{t\to\pm\infty} f(t)e^{-\alpha(t)} = 0$, with the $\lim_{t\to\pm\infty} \alpha(t)/|t| = \infty$. Deng also obtained a result concerning the closure of the linear span of $\{e^{\lambda_n t}\}$ when it is incomplete. For similar frequencies λ_n, results for the completeness of the system $\{e^{\lambda_n t}\}$ in a weighted $L^2(-\infty, +\infty)$ space have been derived by Vinnitskii and Shapovalovskii [22,21]. We also note, that in case the λ_n satisfy $n/|\lambda_n| \mapsto \infty$, results for the completeness and minimality of the system $\{e^{\lambda_n t}\}$ in certain weighted $L^p(-\infty, \infty)$ spaces are found in the survey papers of Sedletskii [18,19].

In the spirit of all these results, the aim of this article is to study the approximation properties of the *multiplicity* exponential system E_Λ

$$E_\Lambda = \{t^k e^{\lambda_n t} : k = 0, 1, 2, \ldots, \mu_n - 1\}_{n=1}^{\infty}, \tag{1.1}$$

in some weighted Banach space of continuous functions on $\mathbf{R}$, and in some weighted $L^p(-\infty, \infty)$ spaces of measurable functions, with $p \in [1, \infty)$. The spaces are denoted by C_w and L^p_w (see Definition 1.2). The system E_Λ is associated with the real multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$, that is, a sequence where $\{\lambda_n\}$ are distinct positive real numbers satisfying $0 < \lambda_n < \lambda_{n+1}$ and increasing to infinity, and where each λ_n appears μ_n times with μ_n not necessarily bounded. The sequence Λ has density D counting multiplicities, that is, $\lim_{t\to\infty} n_\Lambda(t)/t = D$, where $n_\Lambda(t) = \sum_{\lambda_n \leqslant t} \mu_n$ is the counting function of Λ.

We now describe the weighted Banach spaces that we denote by C_w and L^p_w. First, we introduce a class of functions that we denote by $\mathcal{A}_{\rho,\tau}$.

Definition 1.1. The elements of class $\mathcal{A}_{\rho,\tau}$ are all the real-valued, non-negative continuous functions $w(t)$ defined on the real line **R** that satisfy the following properties:

(i) $w(t)$ is a convex function on the real line **R**, $w(0) = 0$, and

$$w(t) \geqslant t^2 \quad \forall\, t \geqslant \tau \geqslant 0. \tag{1.2}$$

(ii) For all $A > 0$ there is a positive number $t(A)$ such that

$$w(t + A) \geqslant w(t) + t \quad \forall\, t \geqslant t(A). \tag{1.3}$$

(iii) For $t < 0$, there is some $\rho > 0$ so that

$$w(t) \leqslant \rho |t|. \tag{1.4}$$

We note that we use the symbol $\mathcal{A}_{\rho,\tau}$ because of the constants ρ and τ that appear in (1.2) and (1.4). An example of $w \in \mathcal{A}_{\rho,\tau}$ is

Example 1.1.

$$w(t) = \begin{cases} e^t - 1, & t \geqslant 0, \\ |t|, & t < 0. \end{cases} \tag{1.5}$$

Definition 1.2. For a function $w(t) \in \mathcal{A}_{\rho,\tau}$, we denote by C_w the set of all complex-valued continuous functions $f(t)$ defined on the real line **R**, satisfying the condition $\lim_{|t|\to\infty} f(t)e^{-w(t)} = 0$. Thus, the $\sup\{|f(t)e^{-w(t)}| : t \in R\}$ is finite, and we define the norm of f as

$$\|f\|_{C_w} = \sup\{|f(t)e^{-w(t)}| : t \in R\}.$$

For a function $w(t) \in \mathcal{A}_{\rho,\tau}$, and $p \in [1, \infty)$, we denote by L^p_w the set of all complex-valued measurable functions $f(t)$ defined on the real line **R**, such that $\int_{-\infty}^{+\infty} |f(t)e^{-w(t)}|^p\, dt < \infty$. The norm of f is defined as

$$\|f\|_{L^p_w} = \left(\int_{-\infty}^{+\infty} |f(t)e^{-w(t)}|^p\, dt \right)^{\frac{1}{p}}.$$

Thus, C_w and L^p_w equipped with these norms become weighted Banach spaces.

One deduces that the **span** of the exponential system E_Λ as defined in (1.1), is a linear subspace of C_w and L^p_w. Thus, it is natural to ask whether E_Λ is complete in these spaces in their respective norms, that is, whether $C_w = \overline{span}(E_\Lambda)$ in $\| \, \|_{C_w}$ and $L^p_w = \overline{span}(E_\Lambda)$ in $\| \, \|_{L^p_w}$. In other words, we ask if for each $f \in C_w(L^p_w)$ and any $\varepsilon > 0$, there is an exponential polynomial $\sum_{n=1}^{k} P_n(t)e^{\lambda_n t}$, where $P_n(t)$ is a polynomial of t of degree at most $\mu_n - 1$, such that

$$\left\| f(t) - \sum_{n=1}^{k} P_n(t)e^{\lambda_n t} \right\|_{C_w(L^p_w)} < \varepsilon.$$

For example, consider a polynomial $Q_n(t) = \sum_{k=1}^{n} a_k t^k$ which belongs to C_w with $w(t)$ as in Example 1.1. Does $Q_n \in \overline{span}(E_\Lambda)$ in $\| \, \|_{C_w}$?

Our main result (Theorem 1.1) in this article, is the derivation, for a certain class of multiplicity sequences, of necessary and sufficient conditions in order for E_Λ to be complete or incomplete in the weighted Banach spaces C_w and L^p_w. Moreover, we prove that if E_Λ is incomplete in the weighted spaces, then for every function f in the closure of the linear span of E_{Λ^*} in $\| \, \|_{C_w}$ (or in $\| \, \|_{L^p_w}$), where $E_{\Lambda^*} = \{t^{\mu_n-1}e^{\lambda_n t}\}_{n=1}^{\infty}$, there is an entire function $g(z)$ represented by a Taylor–Dirichlet series (Theorem 1.3) so that $g(x) = f(x)$ for all (or almost all) $x \in \mathbf{R}$. Here, and throughout the rest of this article, by the phrase $g(x) = f(x)$ *for almost all* $x \in \mathbf{R}$, we mean that $g(x) = f(x)$ for all $x \in \mathbf{R}$ with the possible exception on some set $A \subset \mathbf{R}$ of Lebesgue measure zero. We also prove (Theorem 1.4) that E_Λ is incomplete in C_w or in L^p_w, if and only if it is minimal in that space. The system E_Λ is called minimal in $\| \, \|_{C_w}$ (in $\| \, \|_{L^p_w}$), if every element $t^k e^{\lambda_n t}$ of E_Λ cannot be approximated by the rest of the elements, that is, it does not belong to the closure of the linear span of $E_\Lambda \backslash \{t^k e^{\lambda_n t}\}$ in $\| \, \|_{C_w}$ (in $\| \, \|_{L^p_w}$).

1.1. Multiplicity sequences

We remark that many ideas and auxiliary results that we shall use, are based on the work done in [23]. Such is the multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ which is constructed from a sequence $\mathbf{A}=\{a_n\}$ which belongs to the class $\mathbf{L(c, D)}$. This is the class of all increasing, real, positive sequences $\mathbf{A}=\{a_n\}$, which satisfy for some positive constant c (see also [23]) the spacing condition $a_{n+1} - a_n \geqslant c$ for all $n \geqslant 1$, and have density D, that is, the $\lim_{n\to\infty} \frac{n}{a_n} = D \geqslant 0$. The multiplicity sequence is constructed as follows:

Definition 1.3. Let the sequence $\mathbf{A} \in \mathbf{L(c, D)}$ and α, β real positive numbers so that $\alpha + \beta < 1$. We say that a sequence $\mathbf{B} = \{b_n\}_{n=1}^{\infty}$ with real positive terms b_n, not necessarily in an increasing order, belongs to the class $\mathbf{A}_{\alpha,\beta}$ if for all $n \in \mathbf{N}$ we have

$$b_n \in \{z : |z - a_n| \leqslant a_n^{\alpha}\}, \tag{1.6}$$

and for all $m \neq n$ one of the following holds:

(i) $b_m = b_n$.
(ii) $|b_m - b_n| \geqslant \max\{e^{-a_m^{\beta}}, e^{-a_n^{\beta}}\}$.

One observes that (i) allows for the sequence $\mathbf{B}$ to have coinciding terms. Also note that (ii) allows for non-coinciding terms to come *very close* to each other. We may now rewrite $\mathbf{B}$ in the form of a multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$, by grouping together all those terms that have the same modulus, and ordering them so that $\lambda_n < \lambda_{n+1}$. We shall call this form of $\mathbf{B}$ the (λ, μ) **reordering** (see also [23]).

In the following example of a multiplicity sequence, conditions (i)–(ii) of Definition 1.3 allow for the multiplicities μ_n to tend to infinity, and furthermore for the relation $\liminf_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0$ to hold.

Example 1.2. Let

$$\lambda_n = \begin{cases} \left(\dfrac{n+1}{2}\right)^2, & n \text{ odd}, \\ \dfrac{n^2}{4} + \dfrac{4}{n^2}, & n \text{ even}, \end{cases} \qquad \mu_n = \begin{cases} \dfrac{n+3}{2}, & n \text{ odd}, \\ \dfrac{n}{2}, & n \text{ even}. \end{cases} \tag{1.7}$$

Then, $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ is the (λ, μ) **reordering** of some sequence constructed from the sequence of natural numbers **N**.

Proof. We start with the sequence of natural integers **N** which is an $\mathbf{L}(\mathbf{1}, \mathbf{1})$ sequence, and write **N** as $\{n^2 + k : k = 0, 1, 2, \ldots, 2n\}_{n=1}^{\infty}$. Then, for all $n \in \mathbf{N}$, shift all the terms $n^2 + k$ for $k \in \{0, 1, 2, \ldots, n\}$ to the number n^2, and all the terms $n^2 + k$ for $k \in \{n, n+1, \ldots, 2n\}$ to the number $n^2 + \frac{1}{n^2}$. By simple calculations, we can prove that this shifting yields a sequence **B** which is an $\mathbf{N}_{\alpha,\beta}$ sequence. Its (λ, μ) reordering is just the sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ in (1.7). $\square$

Another interesting example of a multiplicity sequence is the following.

Example 1.3. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the increasing sequence of prime numbers, that is, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5, \ldots$. Let μ_n denote the number of elements in the set $P_n = \{\lambda_n, \lambda_n + 1, \ldots, \lambda_{n+1} - 1\}$. Then the multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ is the (λ, μ) **reordering** of some sequence constructed from the sequence of natural numbers **N**.

Proof. In [2], Baker, Harman and Pintz proved that the relation $\lambda_{n+1} - \lambda_n < \lambda_n^{0.525}$ is satisfied finally for all primes. Due to this estimate, we may start with the sequence of natural integers **N** which is an $\mathbf{L}(\mathbf{1}, \mathbf{1})$ sequence, and by simple calculations, we can prove that shifting all the elements of the set P_n to the prime λ_n, yields a sequence **B** which is an $\mathbf{N}_{\alpha,\beta}$ sequence. Its (λ, μ) reordering is just the sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$. $\square$

We state now two lemmas that were proved in [23] (Lemmas 3.1 and 3.3), regarding multiplicity sequences. They shall be used later on.

Lemma 1.1. *Let* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ and $\mathbf{\Lambda} = \{\lambda_n, \mu_n\}$ *be its* (λ, μ) *reordering. Then for any* $n \in \mathbf{N}$ *one has* $\mu_n \leqslant \chi \lambda_n^{\alpha}$ *where* χ *is some positive constant, independent of* n.

Lemma 1.2. *Let* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ *and* $\mathbf{B} \in A_{\alpha,\beta}$. *Then the* $\lim_{r\to\infty} \frac{n_B(r)}{r} = D$, *where* $n_B(r) = \sum_{b_n \leqslant r} 1$, *that is, the counting function of* **B**.

We remark that if $\Lambda = \{\lambda_n, \mu_n\}$ is the (λ, μ) reordering of the sequence $\mathbf{B} \in A_{\alpha,\beta}$ in the above lemma, it follows that the $\lim_{r\to\infty} \frac{n_\Lambda(r)}{r} = D$ as well.

Define now

$$\sigma_\Lambda(r) = 2 \sum_{\lambda_n \leqslant r} \frac{\mu_n}{\lambda_n}. \tag{1.8}$$

Assuming that $\lambda_1 \geqslant \kappa > 0$, one has that

$$\sigma_\Lambda(r) = 2 \int_\kappa^r \frac{dn_\Lambda(t)}{t} = 2 \frac{n_\Lambda(t)}{t} \Big|_\kappa^r + 2 \int_\kappa^r \frac{n_\Lambda(t)}{t^2}\, dt,$$

and since the $\lim_{r\to\infty} \frac{n_\Lambda(r)}{r} = D$, it follows that

$$\lim_{r\to\infty} \frac{\sigma_\Lambda(r)}{\log r} = 2D. \tag{1.9}$$

1.2. The theorems

We are now ready to give the necessary and sufficient conditions in order for E_Λ to be complete or incomplete in C_w and in L^p_w, which is the main result of this article.

Theorem 1.1. *Let $w(t)$ be a function which belongs to the class $\mathcal{A}_{\rho,\tau}$, and let the sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$. Let the sequence $\mathbf{B} \in A_{\alpha,\beta}$ and $\Lambda = \{\lambda_n, \mu_n\}$ be its (λ, μ) reordering. Then the system E_Λ is incomplete in C_w and in L^p_w for all $p \in [1, \infty)$, if and only if there exists $\eta \in \mathbf{R}$ such that*

$$\int_1^{+\infty} \frac{w(\sigma_\Lambda(t) - \eta)}{1 + t^2}\, dt < \infty. \tag{1.10}$$

Remark 1.1. If $\sigma_\Lambda(r)$ is bounded then E_Λ is always incomplete in C_w and in L^p_w for all $p \in [1, \infty)$, since (1.10) is valid for $\eta = 0$.

Combining (1.9) with our main theorem yields the following.

Theorem 1.2. *For $D > 0$, let the sequences $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ and $\Lambda = \{\lambda_n, \mu_n\}$ be its (λ, μ) reordering. Let also $w_1(t)$ and $w_2(t)$ be functions which belong to the class $\mathcal{A}_{\rho,\tau}$ and satisfy for all $t \geqslant 0$ the relations*

$$w_1(t) \geqslant e^{\gamma t}, \quad \gamma > \frac{1}{2D}, \tag{1.11}$$

$$w_2(t) \leqslant e^{\gamma t}, \quad \gamma < \frac{1}{2D}. \tag{1.12}$$

Then the system E_Λ is complete in the spaces C_{w_1} and $L^p_{w_1}$ for all $p \in [1, \infty)$, and incomplete in C_{w_2} and in $L^p_{w_2}$ for all $p \in [1, \infty)$.

Proof. From relations (1.9) and (1.11) one has for all $\eta \in \mathbf{R}$ that $w_1(\sigma_\Lambda(t) - \eta) \geqslant t$ for $t > t_0$. Thus the integral in (1.10) diverges which implies that E_Λ is complete in C_{w_1} and in $L^p_{w_1}$ for all $p \in [1, \infty)$. Similarly one gets that $w_2(\sigma_\Lambda(t) - \eta) \leqslant t^\kappa$ for $\kappa < 1$, thus the integral in (1.10) converges which implies the incompleteness of E_Λ in C_{w_2} and in $L^p_{w_2}$ for all $p \in [1, \infty)$. $\square$

From the theorem above, since the system E_Λ is not complete in C_{w_2} and in $L^p_{w_2}$, it is natural to investigate the closure of the linear span of E_Λ in the relative norms. At this point the problem is left open, however we deduced a result for the $\overline{span}(E_{\Lambda^*})$ where

$$E_{\Lambda^*} = \{t^{\mu_n - 1} e^{\lambda_n t}\}_{n=1}^{\infty}. \tag{1.13}$$

Theorem 1.3. *Let the sequences $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ and $\Lambda = \{\lambda_n, \mu_n\}$ be its (λ, μ) reordering. Let also $w(t)$ be a function which belongs to the class $\mathcal{A}_{\rho,\tau}$. Assume that the system E_Λ is not complete in C_w or in some L^p_w, $p \in [1, \infty)$. Then the following three results hold*:

(A1) *If the* $\lim_{r\to\infty} \sigma_\Lambda(r) = \infty$, *or*

(A2) *if the* $\lim_{r\to\infty} \sigma_\Lambda(r) < \infty$ *and w belongs to the class Ω where*

$$\Omega = \{w(t) : w(t) \leqslant E_k(t)\ \forall\, t > t_0 \text{ for some } k \geqslant 1\}, \tag{1.14}$$

and the E_k are k-compositions of exponentials, that is,

$$E_k(t) = \underbrace{\exp \circ \exp \circ \cdots \circ \exp}_{k\text{-}times}(t), \tag{1.15}$$

then each function $f \in \overline{span}(E_{\Lambda^*})$ in $\| \|_{C_w}$, can be extended to an entire function $g(z)$ represented by a Taylor–Dirichlet series

$$g(z) = \sum_{n=1}^{\infty} c_n z^{\mu_n - 1} e^{\lambda_n z}, \quad c_n \in \mathbf{C}. \tag{1.16}$$

(B) *If the density D is positive and the function $w(t)$ satisfies for all $t \geqslant 0$ the relation*

$$w(t) \geqslant e^{\delta t}, \quad \delta > 0, \tag{1.17}$$

then for each function $f \in \overline{span}(E_{\Lambda^*})$ in $\| \|_{L^p_w}$, there is an entire function $h(z)$ admitting a representation as in (1.16), *so that $f(x) = h(x)$ for almost all $x \in \mathbf{R}$.*

In fact, we will show that the coefficients c_n in (1.16) are equal to $V_n(f)$ or $S^p_n(f)$ where V_n and S^p_n are bounded linear functionals on C_w and L^p_w, respectively. One can compare the above results with problems of closure of the linear span of the system $\{t^{\lambda_n}\}$ for the spaces $C[0, 1]$ and $L^p(0, 1)$, as treated in [7].

Although we did not get a result concerning the closure of the linear span of E_Λ in $\| \|_{C_w}$ nor in $\| \|_{L^p_w}$, we derive the following theorem regarding the minimality of E_Λ.

Theorem 1.4. *Let $w(t)$ be a function which belongs to the class $\mathcal{A}_{\rho,\tau}$, and let the sequence* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$. *Let the sequence $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ and $\Lambda = \{\lambda_n, \mu_n\}$ be its (λ, μ) reordering. Then, E_Λ is a minimal system in C_w if and only if it is incomplete in C_w. Similarly for L^p_w.*

Finally, we got a result for the completeness of an exponential system in the Banach spaces C_{ww} and L^p_{ww}, defined in a manner similar to C_w and L^p_w. First, we introduce a class of functions that we denote by $\mathcal{B}$.

Definition 1.4. The elements of class $\mathcal{B}$ are all the real-valued, non-negative continuous functions $w(t)$ defined on the real line $\mathbf{R}$, so that $w(t)$ is a convex function on $\mathbf{R}$, $w(0) = 0$, and $w(t) \geqslant t^2$ for all $t \in \mathbf{R}$. Moreover, for all $A > 0$ there is a positive number $t(A)$ such that $w(t + A) \geqslant w(t) + t$ for all $t \geqslant t(A)$, and $w(t - A) \geqslant w(t) + |t|$ for all $t \leqslant -t(A)$.

One observes that if $w \in \mathcal{B}$, then for $t < 0$, $w(t)$ is not necessarily bounded above by $\rho|t|$, as it was the case for $w \in \mathcal{A}_{\rho,\tau}$. An example of a function $w \in \mathcal{B}$ is $w(t) = e^{e^{|t|}} - e$.

Definition 1.5. For a function $w(t) \in \mathcal{B}$, we denote by C_{ww} the set of all complex-valued continuous functions $f(t)$ defined on the real line $\mathbf{R}$ that satisfy the condition $\lim_{|t|\to\infty} f(t)e^{-w(t)} = 0$, and we define the norm of f as $\|f\|_{C_{ww}} = \sup\{|f(t)e^{-w(t)}| : t \in R\}$. Also for $p \in [1, \infty)$ we denote by L^p_{ww} the set of all complex-valued measurable functions $f(t)$ defined on $\mathbf{R}$, such that $\int_{-\infty}^{+\infty} |f(t)e^{-w(t)}|^p \, dt < \infty$. The norm of f is defined as $\|f\|_{L^p_{ww}} = (\int_{-\infty}^{+\infty} |f(t)e^{-w(t)}|^p \, dt)^{\frac{1}{p}}$. Thus C_{ww} and L^p_{ww} equipped with these norms become weighted Banach spaces.

One deduces that the span of the exponential system E_Λ as defined in (1.1) is a linear subspace of C_{ww} and L^p_{ww}. However, this time we will approximate any function $f \in C_{ww}$ or in L^p_{ww}, with a system that includes **negative** frequencies as well. Thus, consider a real multiplicity sequence $\Lambda' = \{\lambda'_j, \mu'_j\}$, with $\lambda'_j > 0$, and denote by $E_{-\Lambda'}$ the system

$$E_{-\Lambda'} = \{t^k e^{-\lambda'_j t} : k = 0, 1, 2, \ldots, \mu'_j - 1\}_{j=1}^{\infty}.$$

We give, without proof, the following sufficient condition in order for the system $E_\Lambda \cup E_{-\Lambda'}$ to be complete in C_{ww} and in L^p_{ww}.

Theorem 1.5. *Let $w(t)$ be a function which belongs to the class $\mathcal{B}$, and let the sequences $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{A}' \in \mathbf{L}(\mathbf{c}', \mathbf{D}')$. Let the sequences $\mathbf{B} \in A_{\alpha,\beta}$, $\mathbf{B}' \in A'_{\alpha',\beta'}$ and $\Lambda = \{\lambda_n, \mu_n\}$, $\Lambda' = \{\lambda'_n, \mu'_n\}$ be their respective re-orderings. If both integrals*

$$I_+ = \int_1^{+\infty} \frac{w(\sigma_\Lambda(t) - \eta)}{1+t^2}\, dt, \qquad I_- = \int_{-\infty}^{-1} \frac{w(-\sigma_{\Lambda'}(|t|) + \eta)}{1+t^2}\, dt,$$

diverge for all $\eta \in \mathbf{R}$, then the system $E_\Lambda \cup E_{-\Lambda'}$ is complete in C_{ww} and in L^p_{ww} for all $p \in [1, \infty)$.

1.3. The main tool

The crucial result which enables us to derive our theorems is Lemma 1.4 which is a variation of the following result due to Fuchs [4, p. 159].

Lemma 1.3. *Let the sequence $A = \{a_n\}_{n=1}^{\infty}$ with $0 < a_n < a_{n+1} \mapsto \infty$ and $a_{n+1} - a_n \geqslant c > 0$ for all $n \geqslant 1$. Then the function*

$$F(z) = \prod_{n=1}^{\infty} \left(\frac{a_n - z}{a_n + z} \right) e^{\frac{2z}{a_n}} \tag{1.18}$$

is analytic in the right half-plane $\mathbf{C}_+ = \{z = x + iy : x > 0\}$, and has simple zeros. With $r = |z|$ and $x = \Re z$, there is some constant $\kappa > 0$ so that the following inequality holds:

$$|F(z)| \leqslant \exp\{x\sigma_A(r) + \kappa x\}, \qquad z \in \mathbf{C}_+. \tag{1.19}$$

Furthermore, for all $z \in \mathbf{C}_+$ that satisfy $|z - a_n| \geqslant c/4$ for all $n \in \mathbf{N}$, one has

$$|F(z)| \geqslant \exp\{x\sigma_A(r) - \kappa x\}. \tag{1.20}$$

We remark that the above lemma played a key role in the deduction of Malliavin's [15] necessary and sufficient conditions, as well as in Anderson and Binmore's [1] result. Since here we are dealing with a multiplicity sequence $\Lambda = \{\lambda_n, \mu_n\}$, a natural analogue of the function $F(z)$ would be a meromorphic function $G(z)$ which vanishes exactly on Λ, with similar bounds as $F(z)$ has in (1.19) and (1.20). The desired function $G(z)$ appears in Lemma 1.4. In order to state this lemma, we need to introduce the following systems of unions of open disks, given the sequences $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ (see also [23]):

$$S_1 = \bigcup_{n=1}^{\infty} B\left(a_n, \frac{e^{-a_n^{\beta}}}{3}\right), \quad S_2 = \bigcup_{n=1}^{\infty} B\left(b_n, \frac{e^{-a_n^{\beta}}}{3}\right), \tag{1.21}$$

$$S_3 = \bigcup_{n=1}^{\infty} B\left(-a_n, \frac{e^{-a_n^{\beta}}}{3}\right), \quad S_4 = \bigcup_{n=1}^{\infty} B\left(-b_n, \frac{e^{-a_n^{\beta}}}{3}\right), \tag{1.22}$$

where as usual

$$B(z_0, r) = \{z : |z - z_0| < r\}.$$

Observe that the disks in S_2 and S_4 are not necessarily disjoint, since for fixed n we might have $b_n = b_m$ for $m \neq n$. Nevertheless, if for fixed n, Γ_n is the set of all integers j so that $b_n = b_j$, that is,

$$\Gamma_n = \{j : b_n = b_j\}, \tag{1.23}$$

then

$$\bigcup_{j \in \Gamma_n} B\left(b_j, \frac{e^{-a_j^{\beta}}}{3}\right) = B\left(b_n, \frac{e^{-a_{l_n}^{\beta}}}{3}\right), \quad l_n = \min\{m : m \in \Gamma_n\}, \tag{1.24}$$

since in this case $e^{-a_j^{\beta}} \leqslant e^{-a_{l_n}^{\beta}}$ for all $j \in \Gamma_n$. Relation (1.24) implies that S_2 and S_4 can be rewritten as infinite unions of non-overlapping disks. Also note, that if $j \in \Gamma_n$ then $\Gamma_j = \Gamma_n$.

Our version of Fuchs Lemma 1.3 is the following:

Lemma 1.4. *Let the sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, the sequence $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ and $\Lambda = \{\lambda_n, \mu_n\}$ be its (λ, μ) reordering. Then the function*

$$G(z) = \prod_{n=1}^{\infty} \left(\frac{b_n - z}{b_n + z}\right) e^{\frac{2z}{b_n}} = \prod_{n=1}^{\infty} \left(\frac{\lambda_n - z}{\lambda_n + z}\right)^{\mu_n} e^{\frac{2z\mu_n}{\lambda_n}} \tag{1.25}$$

is analytic in the right half-plane $\mathbf{C}_+ = \{z = x + iy : x > 0\}$, and vanishes exactly on the sequence $\Lambda = \{\lambda_n, \mu_n\}$. With $r = |z|$ and $x = \Re z$, the following inequalities hold for some constants A_1, A_2 and A_3:

$$|G(z)| \leqslant \exp\{x\sigma_\Lambda(r) + A_1 x\}, \quad z \in \mathbf{C}_+, \tag{1.26}$$

$$|G(z)| \geqslant \exp\{x\sigma_\Lambda(r) - A_2 x\}, \quad z \in C_+ \cap S_2^c, \tag{1.27}$$

where S_2^c is the complement of S_2 defined in (1.21), *and*

$$\frac{|G^{(\mu_n)}(\lambda_n)|}{\mu_n!} \geqslant \exp\{\lambda_n \sigma_\Lambda(\lambda_n) - A_3 \lambda_n\}, \tag{1.28}$$

where $G^{(\mu_n)}(\lambda_n)$ is the μ_n^{th} derivative function of $G(z)$ evaluated on λ_n.

We remark that (1.26) and (1.27) are essential tools for the proof of our main result, Theorem 1.1. Furthermore, (1.28) will be very crucial for the deduction of Theorem 1.3. The bound of $G^{(\mu_n)}(\lambda_n)$, will yield a sharp upper bound for the norms of linear functionals on C_w and L^p_w and this will lead us to the result concerning the closure of the $span(E_{\Lambda^*})$ in $\| \|_{C_w}$ and in $\| \|_{L^p_w}$.

The proof of Lemma 1.4 will occupy §3. Some other auxiliary results are also stated in §2, while the proofs of Theorems 1.1, 1.3, and 1.4 are given in §4, §5 and §6, respectively.

2. Notations and auxiliary results

In addition to the spacing condition $a_{n+1} - a_n \geqslant c > 0$, assume that the sequence $A = \{a_n\}_{n=1}^{\infty}$ in Fuchs result, has density D, that is, $n/a_n \mapsto D$. Then with $F(z)$ as in Lemma 1.3, the function $G(z)$ in Lemma 1.4 can also be written as

$$G(z) = F(z)M_1(z)M_2(z)e^{\xi z}, \tag{2.1}$$

where

$$M_1(z) = \prod_{n=1}^{\infty} \left(\frac{1 - \dfrac{z}{b_n}}{1 - \dfrac{z}{a_n}} \right), \quad M_2(z) = \prod_{n=1}^{\infty} \left(\frac{1 + \dfrac{z}{a_n}}{1 + \dfrac{z}{b_n}} \right), \tag{2.2}$$

define meromorphic functions and ξ is a real number with $\xi = \sum_{n=1}^{\infty} \left(\frac{2}{b_n} - \frac{2}{a_n} \right)$. The existence of ξ is justified by the fact that the series converges absolutely whenever $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$. For the same reason $M_1(z)$ and $M_2(z)$ are well defined.

We also note that in order to obtain the bounds for G in (1.26) and (1.27), one needs to derive bounds for the meromorphic functions M_1 and M_2. These are given in the following lemma which is a slight variation of a result in [23] (Lemma 3.4).

Lemma 2.1. *Let $M_1(z)$ and $M_2(z)$ be the meromorphic functions as in* (2.2) *and let S_i, $i \in \{1, 2, 3, 4\}$ be the systems of disks defined in* (1.21) *and* (1.22). *With $r = |z|$, for every $\varepsilon > 0$ as $r \mapsto \infty$ one has*

$$|M_1(z)| = O(e^{\varepsilon r}) \quad \text{whenever } z \notin S_1, \tag{2.3}$$

$$\frac{1}{|M_1(z)|} = O(e^{\varepsilon r}) \quad \text{whenever } z \notin S_2. \tag{2.4}$$

Similarly, $|M_2(z)| = O(e^{\varepsilon r})$ and $1/|M_2(z)| = O(e^{\varepsilon r})$, provided z lies outside S_4 and S_3, respectively.

We note that in [23] the meromorphic function is actually an even function. Nevertheless, in our case the upper and lower estimates are deduced the same way.

Another auxiliary result that we shall need is the following.

Lemma 2.2. *Let the sequences $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$. Then the sums*

$$\sigma_{\mathbf{A}}(r) - \sigma_{\mathbf{B}}(r) = \sum_{a_n \leqslant r} \frac{2}{a_n} - \sum_{b_n \leqslant r} \frac{2}{b_n}$$

are uniformly bounded with respect to r.

Proof. We shall separate the terms of the sequences into two groups. Define the sets

$$L_1 = \{a_n, b_n : a_n - a_n^{\alpha} \leqslant b_n < a_n\},\ L_2 = \{a_n, b_n : a_n \leqslant b_n \leqslant a_n + a_n^{\alpha}\},$$

and observe that $a_n \in L_1$ if and only if $b_n \in L_1$ (similarly for L_2). Consider then the sums

$$\sum_1(r) = \sum_{\substack{a_n \leqslant r \\ a_n \in L_1}} \frac{2}{a_n} - \sum_{\substack{b_n \leqslant r \\ b_n \in L_1}} \frac{2}{b_n}, \quad \sum_2(r) = \sum_{\substack{a_n \leqslant r \\ a_n \in L_2}} \frac{2}{a_n} - \sum_{\substack{b_n \leqslant r \\ b_n \in L_2}} \frac{2}{b_n},$$

thus $\sigma_{\mathbf{A}}(r) - \sigma_{\mathbf{B}}(r) = \sum_1(r) + \sum_2(r)$. We will prove that the sum $\sum_1$(r) is uniformly bounded. In a similar way one can prove that $\sum_2$(r) is also uniformly bounded, thus the same holds for their sum $\sigma_{\mathbf{A}}(r) - \sigma_{\mathbf{B}}(r)$.

Since $\{a_n\}$ is an increasing sequence, then for an arbitrary $r > 0$ there is an integer $h(r)$ such that $a_{h(r)} \leqslant r$ and $a_k > r$ for every $k > h(r)$. If $a_n \in L_1$ and $a_n \leqslant r$ then $b_n < r$ as well since $b_n < a_n$. It is however possible that some $b_n \in L_1$ with $b_n \leqslant r$ while $a_n > r$. We note that for any such b_n, one has from the L_1 definition that $b_n > r - r^{\alpha}$. Thus we now express the sum $\sum_1(r)$ as

$$\sum_1(r) = \sum_{\substack{1 \leqslant n \leqslant h(r) \\ b_n \in L_1, a_n \in L_1}} \left(\frac{2}{a_n} - \frac{2}{b_n}\right) - \sum_{\substack{r - r^{\alpha} < b_n \leqslant r \\ b_n \in L_1, n > h(r)}} \frac{2}{b_n}. \tag{2.5}$$

We show now that the two sums on the right side of (2.5) are uniformly bounded. First observe that since $b_n \geqslant a_n/2$ for $n > n_0$, $n/a_n \mapsto D$, and $\alpha < 1$, one gets that

$$\sum_{\substack{1 \leqslant n \leqslant h(r) \\ b_n \in L_1, a_n \in L_1}} \left|\frac{1}{a_n} - \frac{1}{b_n}\right| \leqslant \sum_{n=1}^{\infty} \left|\frac{a_n - b_n}{b_n a_n}\right| \leqslant \sum_{n=1}^{\infty} \frac{a_n^{\alpha}}{b_n a_n} < \infty. \tag{2.6}$$

Second, observe that since $\frac{r}{r - r^{\alpha}} \mapsto 1$ as $r \mapsto \infty$, and $n_B(t)/t \mapsto D$ as $t \mapsto \infty$ (see Lemma 1.2), one gets that

$$\sum_{\substack{r - r^{\alpha} < b_n \leqslant r \\ b_n \in L_1, n > h(r)}} \frac{1}{b_n} \leqslant \int_{r - r^{\alpha}}^{r} \frac{dn_B(t)}{t} < \infty, \tag{2.7}$$

with the last inequality valid after integration by parts. From relations (2.6) and (2.7) we get that the function $\sum_1(R)$ is uniformly bounded. □

Due to the convexity of $w(t)$ on the real line we get the following result.

Lemma 2.3. *Let $w(t)$ be a function which belongs to the class $\mathcal{A}_{\rho,\tau}$. For $x \in \mathbf{R}$ let*

$$w^*(x) = \sup\{xt - w(t) : t \in \mathbf{R}\}, \tag{2.8}$$

that is, the Fenchel transform of the convex function $w(x)$. Then w^* is a nonnegative convex function on the real line, $w^*(0) = 0$ and $\lim_{x \to +\infty} w^*(x)/x = +\infty$. Furthermore, $(w^*)^* = w$.

Proof. Since $w(0) = 0$, one deduces that $w^*(0) = 0$ also. The fact that w^* is a non-negative convex function and satisfies $(w^*)^* = w$ can be found in the book of Rockafellar [17].

We also note that w^* satisfies the relation $\lim_{x\to+\infty} w^*(x)/x = +\infty$ because

$$\frac{w^*(x)}{x} = \frac{\sup\{xt - w(t) : t \in \mathbf{R}\}}{x} \geqslant \frac{xt}{x} - \frac{w(t)}{x} \quad \forall t \in \mathbf{R},$$

and letting $x \mapsto \infty$, shows that the $\liminf_{x\to+\infty} w^*(x)/x \geqslant t$ for all $t \in \mathbf{R}$. $\square$

Finally, we end this section by recalling Malliavin's uniqueness theorem [15] (see also [10,13]), about G.N. Watson's problem for the half-plane.

Lemma 2.4. *Let $\beta(t)$ be a non-negative convex function on the real line* $\mathbf{R}$ *satisfying* (1.2), *and assume that*

$$\beta^*(t) = \sup\{tx - \beta(x) : x \in \mathbf{R}\}$$

is the Fenchel transform of the function $\beta(x)$. Suppose that $\lambda(r)$ is an increasing function on $[0, \infty)$ satisfying for $R > r > 1$ and some positive constant κ

$$\lambda(R) - \lambda(r) \leqslant \kappa(\log R - \log r + 1).$$

If there exists a non-trivial analytic function $f(z)$ in $\mathbf{C}_+$, satisfying

$$|f(z)| \leqslant \exp\{\kappa x + \beta(x) - x\lambda(|z|)\}, \quad z = x + iy \in \mathbf{C}_+,$$

then there exists $\eta \in \mathbf{R}$ such that

$$\int_1^{+\infty} \frac{\beta^*(\lambda(t) - \eta)}{1 + t^2}\, dt < \infty.$$

The section that follows is devoted to the proof of Lemma 1.4, which is the main tool in order to derive our theorems.

3. The function $G(z)$: proof of Lemma 1.4

First we state and prove another lemma using methods similar to the ones as in the proof of Theorem 2.1 in [23].

Lemma 3.1. *Let the sequence* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, *the sequence* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *and* $\Lambda = \{\lambda_n, \mu_n\}$ *be its (λ, μ) reordering. Then with $r = |z|$, $x = \Re z$, and S_2 the set defined in* (1.21), *the function $G(z)$ in* (1.25) *satisfies for some constant A_2 the relation*

$$|G(z)| \geqslant \exp\{x\sigma_\Lambda(r) - A_2 x\}, \quad z \notin S_2, \tag{3.1}$$

in the region $\Upsilon = \{z : r < 12x\}$.

Proof. Since the zeros and poles of the meromorphic function $M_2(z)$ as defined in (2.2) lie in the left half-plane $C_- = \{z : \Re z < 0\}$, from Lemma 2.1 we get that

$$\frac{1}{|M_1(z)||M_2(z)|} = O(e^{\varepsilon r}), \quad z \in C_+ \cap S_2^c.$$

Then from (2.1), Lemma 1.3 and the relation above, we get that

$$|G(z)| \geqslant \exp\{x\sigma_A(r) - \kappa x - \varepsilon r + \xi x\}, \tag{3.2}$$

provided $z \notin S_2$ and $|z - a_n| \geqslant c/4$ for all $n \in \mathbf{N}$.

Since $r < 12x$ and the sums $\sigma_A(r) - \sigma_\Lambda(r)$ are uniformly bounded (Lemma 2.2), then there is some real constant A_2 so that instead of (3.2) we have

$$|G(z)| \geqslant \exp\{x\sigma_\Lambda(r) - A_2 x\}, \tag{3.3}$$

provided $z \notin S_2$ and $|z - a_n| \geqslant c/4$ for all $n \in \mathbf{N}$.

Observe that in order to prove (3.1) when $r < 12x$, it remains to remove the condition $|z - a_n| \geqslant c/4$ for all $n \in \mathbf{N}$ from (3.3). To this end, suppose that $\{z_k\}$ is a sequence of complex numbers so that for any $k \in \mathbf{N}$ we have $|z_k - a_k| < c/4$ and $|z_k - b_n| \geqslant e^{-|a_n|^\beta}/3$ for all $n \in \mathbf{N}$. **The rest** of the proof is devoted in showing that (3.3) holds for $|G(z_k)|$. This makes the condition $|z - a_n| \geqslant c/4$ for all $n \in \mathbf{N}$ redundant, thus proving (3.1).

Take any $k \in \mathbf{N}$. Observe that for an arbitrary $z \in \partial B(a_k, c/4)$ the relation $|z - b_n| \geqslant e^{-|a_n|^\beta}/3$ might not hold for all $n \in \mathbf{N}$. Thus, we consider the larger closed disk $\overline{B}(a_k, c/3)$ and claim that there is a constant $\tau \in (c/4, c/3)$ so that for all $z \in \partial B(a_k, \tau)$, one has $|z - b_n| \geqslant e^{-|a_n|^\beta}/3$ for all $n \in \mathbf{N}$. In other words, every point on this circle satisfies this spacing condition. This claim was justified by the author in [23] (see the proof of Theorem 2.1). In that section it was also proved that the number of b_n terms in the closed disc $\overline{B}(a_k, \tau)$, denoted by $\#\left(\{b_n : b_n \in \overline{B}(a_k, \tau)\}\right)$, satisfies the relation

$$\#\left(\{b_n : b_n \in \overline{B}(a_k, \tau)\}\right) \leqslant \frac{4a_k^\alpha}{c}. \tag{3.4}$$

Assume now that $G(z)$ has zeros in the closed disk $\overline{B}(a_k, \tau)$, and define $Y_k = \{n : b_n \in \overline{B}(a_k, \tau)\}$. Then write

$$G(z) = \prod_{n \in Y_k} \left(\frac{b_n - z}{b_n + z}\right) e^{\frac{2z}{b_n}} \prod_{n \notin Y_k} \left(\frac{b_n - z}{b_n + z}\right) e^{\frac{2z}{b_n}}. \tag{3.5}$$

For all $n \in Y_k$ since $|z_k - a_k| \leqslant \tau$ and $|b_n - a_k| \leqslant \tau$, then one has that $|z_k| \approx |a_k| \approx |b_n|$. Also one has that $|a_n| \leqslant 2|a_k|$ for all $n \in Y_k$. Combining all these with $|z_k - b_n| \geqslant e^{-|a_n|^\beta}/3$, yields for all $n \in Y_k$

$$\left|\frac{b_n - z_k}{b_n + z_k}\right| \geqslant \frac{e^{-a_n^\beta}}{9r_k} \geqslant \frac{e^{-2a_k^\beta}}{9r_k} \geqslant e^{-3r_k^\beta}.$$

The last inequality holds since $0 < \beta < 1$. Also one has that $\left|e^{\frac{2z_k}{b_n}}\right| = e^{\frac{2x_k}{b_n}} \geqslant 1$, since $x_k > 0$ and $b_n > 0$. Then, combining (3.4) with all these inequalities and the fact that z is in the region Υ, yields

$$\left|\prod_{n \in Y_k} \left(\frac{b_n - z_k}{b_n + z_k}\right) e^{\frac{2z_k}{b_n}}\right| \geqslant \left(e^{-3r_k^\beta}\right)^{\frac{4r_k^\alpha}{c}} = e^{-12\frac{r_k^{\beta+\alpha}}{c}} \geqslant e^{-\varepsilon r_k} \geqslant e^{-12\varepsilon x_k}. \tag{3.6}$$

Next, we note that for all $n \in Y_k$ and all $z \in \partial B(a_k, \tau)$ one gets

$$\left|\frac{b_n - z}{b_n + z}\right| \leqslant \frac{\tau}{3r_k} \leqslant e^{-3}, \quad \left|e^{\frac{2z}{b_n}}\right| = e^{\frac{2x}{b_n}} \leqslant e^3.$$

Therefore $\left|\prod_{n\in Y_k}\left(\frac{b_n-z}{b_n+z}\right)e^{\frac{2z}{b_n}}\right| \leqslant 1$. This and (3.5) imply that for all $z \in \partial B(a_k, \tau)$ one has

$$|G(z)| \leqslant \left|\prod_{n\notin Y_k}\left(\frac{b_n - z}{b_n + z}\right)e^{\frac{2z}{b_n}}\right|. \tag{3.7}$$

Observe that the product $\prod_{n\notin Y_k}$ in (3.5) has no zeros in the closed disk $\overline{B}(a_k, \tau)$. Thus it takes its minimum value on the boundary. This fact and (3.7) yield

$$\begin{aligned}\left|\prod_{n\notin Y_k}\left(\frac{b_n - z_k}{b_n + z_k}\right)e^{\frac{2z_k}{b_n}}\right| &\geqslant \min_{z\in\partial B(a_k,\tau)}\left|\prod_{n\notin Y_k}\left(\frac{b_n - z}{b_n + z}\right)e^{\frac{2z}{b_n}}\right| \\ &\geqslant \min_{z\in\partial B(a_k,\tau)}|G(z)|.\end{aligned}$$

But for all $z \in \partial B(a_k, \tau)$ relation (3.3) holds since $z \notin S_2$ and $|z - a_n| \geqslant c/4$ for all $n \in \mathbf{N}$. Since $|z_k| \approx |z|$ for all $z \in \partial B(a_k, z)$, one gets that

$$\left|\prod_{n\notin Y_k}\left(\frac{b_n - z_k}{b_n + z_k}\right)e^{\frac{2z_k}{b_n}}\right| \geqslant \exp\{x_k\sigma_\Lambda(r) - A_2x_k\}. \tag{3.8}$$

This relation and (3.6) yield that $|G(z_k)|$ satisfies (3.3). □

We shall also need the following result [4, p. 158].

Lemma 3.2. *Let* $W(z) = \log|(1-z)(1+z)^{-1}e^{2z}|$. *Then* $W(z) \leqslant 2\Re z$ *for* $\Re z \geqslant 0$, *and* $|W(z)| = O(\Re z|z|^2)|$ *for* $|z| \leqslant 1/2$ *and* $\Re z > 0$.

We note that $W(1) = -\infty$, thus the inequality $W(z) \leqslant 2\Re z$ is valid for $z = 1$ as well.

3.1. Proof of Lemma 1.4

Proof of (1.26). We recall that $z = x + iy$ and $|z| = r$. We write $G(z) = \prod_1(z)\prod_2(z)$ where $\prod_1(z)$ contains the terms with $b_n \leqslant 2r$ and $\prod_2(z)$ those with $b_n > 2r$. First we get an upper bound for $\prod_1(z)$. We apply Lemma (3.2) and get

$$\log\left|\prod_1(z)\right| \leqslant \sum_{b_n\leqslant 2r} 2xb_n^{-1} = \sum_{b_n\leqslant r} 2xb_n^{-1} + \sum_{r<b_n\leqslant 2r} 2xb_n^{-1},$$

and

$$\sum_{r<b_n\leqslant 2r} 2xb_n^{-1} = 2x\int_r^{2r}\frac{dn_B(t)}{t} = 2x\left(\frac{n_B(t)}{t}\bigg|_r^{2r} + \int_r^{2r}\frac{n_B(t)}{t}\frac{1}{t}\,dt\right) \leqslant \sigma x$$

for some $\sigma > 0$, since $n_B(t)/t \mapsto D$. Thus we get that

$$\log\left|\prod_1(z)\right| \leqslant x\sigma_\Lambda(r) + \sigma x. \tag{3.9}$$

Next we get an upper bound for $\prod_2(z)$. By Lemma (3.2), one gets that

$$\left|\log\left|\prod_2(z)\right|\right| \leqslant \sum_{b_n>2r} M\frac{xr^2}{b_n^3} \leqslant \frac{Mxr}{2}\sum_{b_n>2r}\frac{1}{b_n^2} = \frac{Mxr}{2}\int_{2r}^{\infty}\frac{dn_B(t)}{t^2}.$$

Since $n_B(t)/t \mapsto D$ we get that

$$\left|\log\left|\prod_2(z)\right|\right| \leqslant \psi x, \quad \psi > 0. \tag{3.10}$$

Combining (3.9) and (3.10), shows that (1.26) holds. □

Proof of (1.27). We consider two cases, one when $r \geqslant 12x$ and another when $r < 12x$.

Case 1 ($r \geqslant 12x$):

We use again (3.10) to get a lower bound for $\prod_2(z)$, that is

$$\log\left|\prod_2(z)\right| \geqslant -\psi x, \quad \psi > 0. \tag{3.11}$$

Then we want to get a lower bound for $|\prod_1(z)|$. First we get that

$$\left|\frac{b_n - z}{b_n + z}\right|^2 = 1 - \frac{4b_n x}{(b_n + x)^2 + y^2} > 1 - \frac{4b_n x}{r^2} \geqslant 1 - \frac{8x}{r} \geqslant e^{-\frac{2Cx}{r}},$$

for some $C > 0$. Since $n_B(r)/r \mapsto D$, then $n_B(2r) < 3rD$. Thus

$$\left|\prod_{b_n \leqslant 2r}\frac{b_n - z}{b_n + z}\right| \geqslant \left(e^{-\frac{Cx}{r}}\right)^{3rD} = e^{-3CDx}.$$

We also have that $\left|\prod_{b_n \leqslant 2r} e^{\frac{2z}{b_n}}\right| = e^{2x\sum_{b_n \leqslant 2r}\frac{1}{b_n}} = e^{x\sigma_\Lambda(2r)} \geqslant e^{x\sigma_\Lambda(r)}$, thus if $r \geqslant 12x$ we have

$$\left|\prod_1(z)\right| \geqslant e^{x\sigma_\Lambda(r) - 3CDx}. \tag{3.12}$$

Combining (3.11) and (3.12), shows that (1.27) holds.

Case 2: ($r < 12x$): Its proof is given in Lemma 3.1. □

Proof of (1.28). Fix some positive integer n and write $G(z) = O_n(z)Q_n(z)$ where

$$O_n(z) = \prod_{k\in\Gamma_n}\left(\frac{b_k - z}{b_k + z}\right)e^{\frac{2z}{b_k}}, \quad Q_n(z) = \prod_{k\notin\Gamma_n}\left(\frac{b_k - z}{b_k + z}\right)e^{\frac{2z}{b_k}}, \tag{3.13}$$

and Γ_n as defined in (1.23). Since $\mathbf{B} = \mathbf{\Lambda} = \{\lambda_n, \mu_n\}$, there is some λ_j so that $\lambda_j = b_n$ with multiplicity μ_j. Thus, we can write

$$G(z) = \left(\frac{\lambda_j - z}{\lambda_j + z}\right)^{\mu_j} e^{\frac{2z\mu_j}{\lambda_j}} Q_n(z).$$

We can also write

$$G(z) = (\lambda_j - z)^{\mu_j} \left[\left(\frac{e^{\frac{2z}{\lambda_j}}}{\lambda_j + z} \right)^{\mu_j} Q_n(z) \right].$$

Denote now by $H(z)$ the function inside the brackets [ ], thus $H(\lambda_j) \neq 0$. Then one deduces that

$$\frac{G^{(\mu_j)}(\lambda_j)}{\mu_j!} = (-1)^{\mu_j} H(\lambda_j) = \left(\frac{-e^2}{2\lambda_j}\right)^{\mu_j} Q_n(\lambda_j). \tag{3.14}$$

We want to obtain a lower bound for $|G^{(\mu_j)}(\lambda_j)/\mu_j!|$. First, note that from Lemma 1.1 we get that $\lambda_j^{\mu_j} \leqslant \lambda_j^{\chi\lambda_j^{\alpha}} \leqslant e^{\lambda_j}$, thus

$$\left(\frac{e^2}{2\lambda_j}\right)^{\mu_j} > \left(\frac{1}{\lambda_j}\right)^{\mu_j} \geqslant e^{-\lambda_j}. \tag{3.15}$$

Next, consider the closed disk $\overline{B}(b_n, e^{-a_{l_n}^{\beta}}/3)$ where l_n is defined in (1.24). We claim that for all z on the boundary of the disk one has $z \notin S_2$ (see (1.21)), in other words we have $|z - b_k| \geqslant e^{-a_k^{\beta}}/3$ for all $k \in \mathbf{N}$.

Indeed, if $b_k \in \Gamma_n$ the relation is trivial. Assume then that $b_k \notin \Gamma_n$ and consider the case when $a_k \leqslant a_{l_n}$. Then from (ii) in Definition 1.3 one has $|b_{l_n} - b_k| \geqslant e^{-a_k^{\beta}}$, and since $|z - b_{l_n}| = e^{-a_{l_n}}$ one gets that $|z - b_k| = |(z - b_{l_n}) + (b_{l_n} - b_k)| \geqslant 2e^{-a_k^{\beta}}/3$. Similarly we treat the case $a_k > a_{l_n}$, and our claim is justified.

Then observe that for all z on the boundary of the disk $\overline{B}(b_n, e^{-a_{l_n}^{\beta}}/3)$ we get that $|G(z)| \leqslant |Q_n(z)|$. This holds because if z is on the boundary and $k \in \Gamma_n$, then for all $n > n_0$ one has that

$$\left|\frac{b_k - z}{b_k + z}\right| e^{\frac{2\Re z}{b_k}} \leqslant \frac{e^{-a_{l_n}^{\beta}}/3}{b_n} e^3 \leqslant \frac{1}{b_n} e^3 < 1.$$

Substitution in (3.13) shows that $|O_n(z)| \leqslant 1$, thus $|G(z)| \leqslant |Q_n(z)|$. This result and the fact that the boundary of the disk $\overline{B}(b_n, e^{-a_{l_n}^{\beta}}/3)$ does not belong to the S_2 system of disks, yield that for all $z \in \partial B(b_n, e^{-a_{l_n}^{\beta}}/3)$, one has that

$$|Q_n(z)| \geqslant |G(z)| \geqslant \exp\{x\sigma_{\Lambda}(r) - A_2 x\} \geqslant \exp\{\lambda_j \sigma_{\Lambda}(\lambda_j) - A_3 \lambda_j\}. \tag{3.16}$$

The second inequality follows from (1.27). The third holds since $\lambda_j \approx |z|$ for all z on the boundary of the disk. Finally, observe that $Q_n(z)$ has no zeros in the closed disk $\overline{B}(b_n, e^{-a_{l_n}^{\beta}}/3)$. Thus it takes its minimum value on the boundary. Combining this with (3.16) gives

$$|Q_n(\lambda_j)| \geqslant \exp\{\lambda_j \sigma_{\Lambda}(\lambda_j) - A_3 \lambda_j\}. \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.14), shows that (1.28) is valid. $\square$

4. Complete exponential systems in C_w and in L_w^p: proof of Theorem 1.1

First we prove the necessity part, that is, the integral in (1.10) converges for some $\eta \in \mathbf{R}$, assuming that E_Λ is incomplete in C_w or in L_w^p with $p \in [1, \infty)$. Second we prove the sufficiency part, that is, if the integral in (1.10) converges for some $\eta \in \mathbf{R}$, then E_Λ is incomplete in C_w and in L_w^p for all $p \in [1, \infty)$. Once more we remark that the meromorphic function $G(z)$ of Lemma 1.4 plays an important role, both for the necessity and for the sufficiency part.

4.1. Necessity for C_w and L_w^p

4.1.1. Necessity for L_w^p

If the system E_Λ is incomplete in L_w^P for some $p \in [1, \infty)$, then by the Hahn–Banach theorem there exists a non-trivial functional T on L_w^P which vanishes on the elements of E_Λ. By the Riesz representation theorem, there exists a function $g(t) \in L^q(-\infty, +\infty)$, where $1/p + 1/q = 1$, so that for every function $f \in L_w^p$ one has that

$$T(f) = \int_{-\infty}^{+\infty} f(t) e^{-w(t)} g(t)\, dt. \tag{4.1}$$

Since T vanishes on the elements of E_Λ, one observes that if for all z in the half-plane $\mathbf{C}_+$ we define the function

$$P(z) = \int_{-\infty}^{+\infty} e^{zt} e^{-w(t)} g(t)\, dt,$$

then $P(z)$ is an analytic function in the half-plane $\mathbf{C}_+$, and from (4.1) one has that $P(z)$ vanishes on λ_n at **least** μ_n times. We show now that for all $z \in \mathbf{C}_+$, $P(z)$ satisfies

$$|P(z)| \leqslant \|g\|_q \left(1 + \frac{1}{x^{\frac{1}{p}}}\right) e^{w^*(x+1)}, \quad \|g\|_q = \left(\int_{-\infty}^{+\infty} |g(t)|^q\, dt\right)^{\frac{1}{q}}. \tag{4.2}$$

First fix some $x > 0$. For $t \geqslant 0$ one has that $e^{xt} e^{-w(t)} = e^{(x+1)t} e^{-w(t)} e^{-t} \leqslant e^{w^*(x+1)} e^{-t} \in L^p(0, \infty)$. Applying the Hölder inequality gives

$$\begin{aligned} \int_0^{+\infty} e^{xt} e^{-w(t)} |g(t)|\, dt &\leqslant \|g\|_q \left(\int_0^{+\infty} e^{pw^*(x+1)} e^{-pt}\, dt\right)^{\frac{1}{p}} \\ &= \|g\|_q e^{w^*(x+1)} \left(\int_0^{+\infty} e^{-pt}\, dt\right)^{\frac{1}{p}} \leqslant \|g\|_q e^{w^*(x+1)}. \end{aligned} \tag{4.3}$$

When $t < 0$, $e^{xt} e^{-w(t)} \leqslant e^{xt} \in L^p(-\infty, 0)$. Applying the Hölder inequality gives

$$\begin{aligned} \int_{-\infty}^{0} e^{xt} e^{-w(t)} |g(t)|\, dt &\leqslant \|g\|_q \left(\int_{-\infty}^{0} e^{pxt}\, dt\right)^{\frac{1}{p}} \\ &\leqslant \|g\|_q \left(\frac{1}{x}\right)^{\frac{1}{p}} \leqslant \|g\|_q \left(\frac{1}{x}\right)^{\frac{1}{p}} e^{w^*(x+1)}. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we have (4.2).

Define now $f(z) = \frac{P(z)}{G(z)}$ where $G(z)$ is the meromorphic function defined in Lemma 1.4. Then $f(z)$ is analytic in the half-plane $\mathbf{C}_+$. By (1.27) and (4.2), one has that

$$|f(z)| \leqslant \left(1 + \frac{1}{x^{\frac{1}{p}}}\right) \|g\|_q \exp\{-x\sigma_\Lambda(r) + A_2 x + w^*(x+1)\}, \quad x > 0, \tag{4.5}$$

provided $z \notin S_2$ with S_2 as defined in (1.21). Since S_2 can be written as a disjoint union of disks whose radii tends to zero, by the Maximus Modulus theorem we get that (4.5) holds for all z in the half-plane $\mathbf{C}_+$. Then, for $x > 1$ one has from (4.5) that

$$|f(z)| \leqslant 2\|g\|_q \exp\{-x\sigma_\Lambda(r) + A_2 x + w^*(x+1)\}, \quad x > 1. \tag{4.6}$$

Let $r(z) = f(z+1)$, thus $r(z)$ is analytic in the half-plane $\{z : \Re z > -1\}$. Then from (4.6) one has that

$$|r(z)| \leqslant 2\|g\|_q \exp\{-(x+1)\sigma_\Lambda(|z+1|) + A_2(x+1) + w^*(x+2)\}, \quad x > 0. \tag{4.7}$$

We claim now that there is a positive constant A_7 so that

$$|r(z)| \leqslant A_7 \exp\{-x\sigma_\Lambda(|z|) + A_7 x + w^*(x)\}, \quad x > 0. \tag{4.8}$$

Assuming this, applying Lemma 2.4 and the relation $(w^*)^* = w$, yields some $\eta \in \mathbf{R}$ such that

$$\int_1^{+\infty} \frac{w(\sigma_\Lambda(t) - \eta)}{1+t^2}\, dt < \infty,$$

and we are done. We now justify our claim. Fix some $x > 0$ and note that since w is convex, then the equation $(x+2)t = w(t)$ has solutions $t = 0$ and some $t_1 > 0$. Since $w(t) \geqslant t^2$ then $t_1 \leqslant x+2$. Let $t_0 \in [0, t_1]$, thus $t_0 \leqslant x+2$ also, be a point so that the $\sup_{t \geqslant 0}\{(x+1)t - w(t)\}$ is achieved there. Then

$$\begin{aligned} w^*(x+2) &= \sup_{t \geqslant 0}\{(x+2)t - w(t)\} = xt_0 - w(t_0) + 2t_0 \\ &\leqslant \sup_{t \geqslant 0}\{xt - w(t)\} + 2x + 4 \leqslant w^*(x) + 2x + 4. \end{aligned} \tag{4.9}$$

Also note that $-(x+1)\sigma_\Lambda(|z+1|) \leqslant -x\sigma_\Lambda(|z|)$ for all $z \in C_+$. Substitution of this result and (4.9) into (4.7) yields (4.8).

4.1.2. *Necessity for C_w*

The proof is similar as for L^p_w.

4.2. *Sufficiency for C_w and L^p_w*

Assume that there exists a real number η such that

$$\int_1^{\infty} \frac{w(\sigma_\Lambda(t) - \eta)}{1+t^2}\, dt < \infty.$$

Our goal is to show that there exist non-trivial bounded linear functionals $T : C_w \mapsto \mathbf{C}$ and $S : L^p_w \mapsto \mathbf{C}$, which vanish on the elements of the exponential system E_Λ. This implies that E_Λ is incomplete in C_w and in L^p_w.

There is some $\delta > 0$ such that $\sigma_\Lambda(t) - \eta \geqslant 0$ for all $t \geqslant \delta$. Then, let $\phi(t)$ be an even function on the real line $\mathbf{R}$ so that

$$\phi(t) = \begin{cases} w(\sigma_\Lambda(t) - \eta), & t \geqslant \delta, \\ w(\sigma_\Lambda(\delta) - \eta), & 0 \leqslant t < \delta. \end{cases}$$

Since $w \in \mathcal{A}_{\rho,\tau}$, by definition $w(t) \leqslant \rho|t|$ for all $t < 0$, for some $\rho > 0$. Then, for this ρ, and since the above integral converges, we consider the Poisson integral of $\phi(t)$

$$u(z) = \frac{x + 2\rho}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(t)}{(x + 2\rho)^2 + (y - t)^2}\, dt,$$

which is a harmonic function in the half-plane $C_{-2\rho} = \{z : \Re z > -2\rho\}$. We then have that for all $z \in C_{-2\rho}$ the following relation holds

$$\begin{aligned} -4u(z) &\leqslant \frac{-4(x + 2\rho)}{\pi} \int_{|t| \geqslant |z+2\rho|} \frac{\phi(t)}{(x + 2\rho)^2 + (y - t)^2}\, dt \\ &\leqslant -\phi(|z + 2\rho|) \\ &= \begin{cases} -w(\sigma_\Lambda(|z + 2\rho|) - \eta), & |z + 2\rho| \geqslant \delta, \\ -w(\sigma_\Lambda(\delta) - \eta), & 0 \leqslant |z + 2\rho| < \delta. \end{cases} \end{aligned} \tag{4.10}$$

Observe now that for all $z \in C_{-2\rho}$ such that $|z + 2\rho| \geqslant \delta$, one has that

$$x(\sigma_\Lambda(|z + 2\rho|) - \eta) - w(\sigma_\Lambda(|z + 2\rho|) - \eta) \leqslant \sup_{t \geqslant 0}\{xt - w(t)\} = w^*(x),$$

thus,

$$-w(\{\sigma_\Lambda(|z + 2\rho|) - \eta\}) \leqslant w^*(x) - x\sigma_\Lambda(|z + 2\rho|) + \eta x + J,$$

where $J > 0$ and large enough, so that we also have $-w(\sigma_\Lambda(\delta) - \eta) \leqslant w^*(x) - x\sigma_\Lambda(|z+2\rho|) + \eta x + J$ for all z such that $|z + 2\rho| < \delta$. Combining these results with (4.10), we get that

$$-4u(z) \leqslant w^*(x) - x\sigma_\Lambda(|z + 2\rho|) + \eta x + J, \quad z \in C_{-2\rho}.$$

Then, if $v(z)$ is the harmonic conjugate of $u(z)$, the function

$$f(z) = e^{-4u(z) - i4v(z)} \tag{4.11}$$

is an analytic function in the half-plane $C_{-2\rho}$, $f(z)$ vanishes nowhere in $C_{-2\rho}$, and satisfies

$$|f(z)| \leqslant e^{w^*(x) - x\sigma_\Lambda(|z+2\rho|) + \eta x + J}, \quad z \in C_{-2\rho}. \tag{4.12}$$

The function $f(z)$ also satisfies

$$|f(x)| \geqslant \sigma e^{-A_5 x} \quad \forall\ x > 1 - 2\rho, \tag{4.13}$$

for some positive constants σ and A_5. Define now the meromorphic function

$$G(z) = \prod_{n=1}^{\infty} \left(\frac{(\lambda_n + 2\rho) - z}{(\lambda_n + 2\rho) + z} \right)^{\mu_n} e^{\frac{2z\mu_n}{\lambda_n + 2\rho}}, \tag{4.14}$$

which is analytic in the half-plane $C_{-2\rho}$, and vanishes exactly on the multiplicity sequence $\Lambda' = \{\lambda_n + 2\rho, \mu_n\}_{n=1}^{\infty}$. We note that $G(z)$ satisfies relation (1.26) as the meromorphic function in (1.25) does, that is,

$$|G(z)| \leqslant \exp\{x\sigma_{\Lambda'}(r) + A_1 x\}, \quad z \in \mathbf{C}_+.$$

Observe now that $\sum_{\lambda_n+2\rho\leqslant r} \frac{\mu_n}{\lambda_n+2\rho} \leqslant \sum_{\lambda_n\leqslant r} \frac{\mu_n}{\lambda_n}$, that is, $\sigma_{\Lambda'}(r) \leqslant \sigma_\Lambda(r)$. Thus

$$|G(z)| \leqslant \exp\{x\sigma_\Lambda(r) + A_1 x\}, \quad z \in \mathbf{C}_+.$$

Next, define $G_\rho(z) = G(z+2\rho)$, thus G_ρ is analytic in the half-plane $C_{-4\rho} = \{z : \Re z > -4\rho\}$, vanishes exactly on the sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$, and has poles $\{-\lambda_n - 4\rho, \mu_n\}_{n=1}^{\infty}$. From the relation above, one has that

$$|G_\rho(z)| \leqslant \exp\{(x+2\rho)\sigma_\Lambda(|z+2\rho|) + A_1(x+2\rho)\}, \quad z \in C_{-2\rho}.$$

Thus, if we combine the above relation with (4.12), this gives

$$\begin{aligned} |G_\rho(z) f(z)| &\leqslant \exp\{w^*(x) + 2\rho\sigma_\Lambda(|z+2\rho|) + A_4 x\} \\ &\leqslant |z+2\rho|^{6\rho D} \exp\{w^*(x) + A_4 x\}, \quad x > -2\rho, \end{aligned}$$

for some real constant A_4. The last inequality follows from (1.9), that is, the relation $\lim_{r\to\infty} \sigma_\Lambda(r) / \log r = 2D$. Next, define

$$g_0(z) = \frac{G_\rho(z) f(z) e^{-2A_4 z}}{(2\rho + 1 + z)^{[6\rho D]+2}},$$

where $[6\rho D]$ is the integral part of $6\rho D$. Then $g_0(z)$ is an analytic function in the half-plane $C_{-2\rho}$, vanishes exactly on the sequence $\Lambda = \{\lambda_n, \mu_n\}$, and satisfies

$$|g_0(z)| \leqslant \frac{\exp\{w^*(x) - A_4 x\}}{1 + |z+2\rho|^2}, \quad x > -2\rho. \tag{4.15}$$

Due to this estimate, by contour integration one deduces that the value of the integral

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(x+iy) e^{-(x+iy)t}\, dy, \quad x > -2\rho, \tag{4.16}$$

does not depend on x, thus it is a function of t only. Therefore, for any fixed $x > -2\rho$, $e^{xt} h_0(t)$ is the Fourier Transform of $g_0(x+iy)$. Our goal is to prove that for all $x > -\rho$, $g_0(x+iy)$ is equal to the Inverse Fourier Transform of $e^{xt} h_0(t)$. In other words,

$$g_0(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) e^{tz}\, dt, \quad x > -\rho. \tag{4.17}$$

For this, it suffices to show that for any fixed $x > -\rho$, $e^{xt} h_0(t)$ belongs to the space $L^1(-\infty, \infty)$. Thus, we need to obtain an upper bound for $|h_0(t)|$.

First recall that for $u \geqslant 0$ one has that the $\sup_{x \geqslant 0}\{ux - w^*(x)\} = w(u)$ (Lemma 2.3). Then for $t \geqslant -A_4$ one gets from (4.16) that

$$\begin{aligned}|h_0(t)| &\leqslant \inf_{x>-2\rho} \exp\{w^*(x) - A_4 x - xt\} \leqslant \inf_{x \geqslant 0} \exp\{w^*(x) - A_4 x - xt\} \\ &= \exp\left\{-\sup_{x \geqslant 0}[(t + A_4)x - w^*(x)]\right\} = \exp\{-w(t + A_4)\} \quad \forall t \geqslant -A_4 \\ &\leqslant \exp\{-w(t) - t\} \quad \forall t \geqslant t(A_4),\end{aligned}$$

where the last inequality follows from (1.3). Since $h_0(t)$ is continuous on $\mathbf{R}$, there is some $\xi > 1$ so that

$$|h_0(t)| \leqslant \xi \exp\{-w(t) - t\} \quad \forall t \geqslant 0. \tag{4.18}$$

We find now an upper bound when $t < 0$. Since for every fixed $x > -2\rho$, $e^{xt}h_0(t)$ is the Fourier Transform of an integrable function, then from the Riemann–Lebesgue Lemma one has that for all $x > -2\rho$, $e^{xt}h_0(t) \mapsto 0$ as $t \mapsto \pm\infty$. Thus for $x = -\frac{3}{2}\rho$, there is some $t_0 > 0$ so that for $t < -t_0$ we have that $|h_0(t)| \leqslant e^{-\frac{3}{2}\rho|t|}$. Since $h_0(t)$ is continuous on $\mathbf{R}$, there is some $\xi' > 1$ so that

$$|h_0(t)| \leqslant \xi' e^{-\frac{3}{2}\rho|t|} \quad \forall t < 0. \tag{4.19}$$

Relations (4.18) and (4.19) imply that for every fixed $x > -\rho$, $e^{xt}h_0(t) \in L^1(-\infty, \infty)$, thus (4.17) is valid. One can also prove that differentiating *k-times* the function $g_0(x)$ with respect to the real variable x, gives

$$g_0^{(k)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) t^k e^{tx}\, dt. \tag{4.20}$$

Since $g(z)$ vanishes on the sequence $\mathbf{\Lambda}$, this implies that

$$\int_{-\infty}^{+\infty} h_0(t) t^k e^{t\lambda_n}\, dt = 0, \quad k \in \{0, \ldots, \mu_n - 1\}, \;\; n = 1, 2, \ldots . \tag{4.21}$$

We define now the bounded linear functionals on C_w and on L_w^p. First, for every function f in the Banach space C_w, let

$$T(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) f(t)\, dt.$$

From (4.18) we get that

$$\int_0^{+\infty} |h_0(t) f(t)|\, dt = \int_0^{+\infty} |h_0(t) e^{w(t)} f(t) e^{-w(t)}|\, dt \leqslant \xi \|f\|_{C_w}, \tag{4.22}$$

and from (4.19) and (1.4) we get that

$$\int_{-\infty}^0 |h_0(t) f(t)|\, dt \leqslant \xi' \|f\|_{C_w} \int_{-\infty}^0 e^{-\frac{\rho|t|}{2}}\, dt = \frac{2\xi'}{\rho} \|f\|_{C_w}. \tag{4.23}$$

From (4.22) and (4.23) one has that T is a bounded linear functional on C_w.

Next, since $w \in \mathcal{A}_{\rho,\tau}$ and (4.18)–(4.19) hold, one deduces that $h_0(t)e^{w(t)} \in L^q(-\infty, +\infty)$ for all $q \geqslant 1$, and furthermore, $h_0(t)e^{w(t)}$ is uniformly bounded on $\mathbf{R}$. Then, for every function f in the Banach spaces L^p_w, $p \in [1, \infty)$, define

$$S(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) f(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (h_0(t)e^{w(t)})(f(t)e^{-w(t)})\, dt.$$

From the Hölder inequality one gets that $|S(f)| \leqslant \kappa \|f\|_{L^p_w}$, $\kappa > 0$, thus S is a bounded linear functional on L^p_w.

Also note that $T(e^{zt}) = g_0(z)$ and $S(e^{zt}) = g_0(z)$ for all $z \in C_+$. Since $g_0(z)$ is not identically zero and vanishes exactly on Λ, then T and S are non-trivial and from (4.21) one has that they vanish on the elements of E_Λ.

5. The closure of the linear span of E_{Λ^*} in C_w and in L^p_w: proof of Theorem 1.3

The main idea in order to prove Theorem 1.3, is to find bounded linear functionals $\{V_n\}_{n=1}^\infty$ on C_w and $\{S^p_n\}_{n=1}^\infty$ on L^p_w, so that if $f_1 \in \overline{span}(E_{\Lambda^*})$ in $\| \, \|_{C_w}$ and $f_2 \in \overline{span}(E_{\Lambda^*})$ in $\| \, \|_{L^p_w}$, the functions

$$g_1(z) = \sum_{n=1}^{\infty} V_n(f_1) z^{\mu_n - 1} e^{\lambda_n z}, \quad g_2(z) = \sum_{n=1}^{\infty} S^p_n(f_2) z^{\mu_n - 1} e^{\lambda_n z},$$

are entire functions with $g_1(x) = f_1(x)$ for all $x \in \mathbf{R}$ and $g_2(x) = f_2(x)$ for almost all $x \in \mathbf{R}$. We remark that the functionals $\{V_n\}$ and $\{S^p_n\}$ are actually extensions, through the Hahn–Banach theorem, of functionals $\{T_n\}$ on the subspace $span(E_{\Lambda^*})$ in C_w and L^p_w (see Lemma 5.2).

The next lemma is the first step in order to find such functionals.

Lemma 5.1. *Let the sequence* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, *the sequence* $\mathbf{B} \in A_{\alpha,\beta}$ *and* $\Lambda = \{\lambda_n, \mu_n\}$ *be its* (λ, μ) *reordering. Let also* $w(t)$ *be a function which belongs to the class* $\mathcal{A}_{\rho,\tau}$. *Assume that the system* E_Λ *is not complete in* C_w *or in some* L^p_w, $p \in [1, \infty)$. *Then, there exist continuous functions* $\{\Psi_n(t)\}_{n=1}^\infty$ *on the real line* $\mathbf{R}$, *so that*

$$\int_{-\infty}^{\infty} \Psi_n(t) t^{\mu_k - 1} e^{\lambda_k t}\, dt = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases} \tag{5.1}$$

Proof. First, fix some positive integer n. Let $G_\rho(z) = G(z+2\rho)$, where $G(z)$ is the meromorphic function as in (4.14), and define

$$G_{\rho,n}(z) = G_\rho(z) \left(\frac{z + \lambda_n + 4\rho}{z - \lambda_n} \right) e^{-\frac{2(z+2\rho)}{\lambda_n + 2\rho}}. \tag{5.2}$$

Then $G_{\rho,n}(z)$ is analytic in the half-plane $C_{-4\rho} = \{z : \Re z > -4\rho\}$ and vanishes exactly $\mu_n - 1$ times at the point λ_n, and μ_k times at the points λ_k, $k \neq n$. One can verify that $G_{\rho,n}(z)$ satisfies (1.26) for the same constant A_1 as $G_\rho(z)$ does.

Since E_Λ is incomplete in C_w or in L^p_w for $p \in [1, \infty)$, it follows from the necessity part of Theorem 1.1 that there exists a real number η such that (1.10) holds. Then, from the proof in the sufficiency part of Theorem 1.1, there exists an analytic function $f(z)$ on $C_{-2\rho}$ with no zeros, such that relation (4.12) is satisfied. Define now

$$\Phi_n(z) = \frac{G_{\rho,n}(z) f(z) e^{-2A_4 z}}{(1+2\rho+z)^{[6\rho D]+2}}. \tag{5.3}$$

It follows that $\Phi_n(z)$ is an analytic function in $C_{-2\rho} = \{z : \Re z > -2\rho\}$, and vanishes exactly $\mu_n - 1$ times at the point λ_n and μ_k times at the points λ_k, $k \neq n$. Since $G_{\rho,n}(z)$ satisfies (1.26) for the same constant A_1 as $G_\rho(z)$ does, then $\Phi_n(z)$ satisfies the relation

$$|\Phi_n(z)| \leqslant \frac{\exp\{w^*(x) - A_4 x\}}{(1+|z+2\rho|^2)}, \quad x > -2\rho, \tag{5.4}$$

as the function $g_0(z)$ in (4.15). From this relation, we deduce by contour integration that the value of the integral

$$H_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi_n(x+iy) e^{-(x+iy)t}\, dy, \quad x > -2\rho, \tag{5.5}$$

does not depend on x, thus it is a function of t only. This function is continuous on $\mathbf{R}$, and since the upper bound in (5.4) is the same as the one in (4.15), there are positive numbers ξ and ξ' equal to the ones in (4.18) and (4.19), and independent of n, so that

$$|H_n(t)| \leqslant \xi \exp\{-w(t) - t\} \quad \forall t \geqslant 0, \tag{5.6}$$

and

$$|H_n(t)| \leqslant \xi' e^{-\frac{3}{2}\rho|t|} \quad \forall t < 0. \tag{5.7}$$

These relations imply that for any fixed $x > -\rho$, $e^{xt} H_n(t) \in L^1(-\infty, \infty)$, thus $\Phi_n(z)$ is equal to the Inverse Fourier Transform of $H_n(t)e^{tx}$, that is,

$$\Phi_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_n(t) e^{tz}\, dt, \quad x > 0. \tag{5.8}$$

Define now

$$\Psi_n(t) = \frac{H_n(t)}{\Phi_n^{(\mu_n - 1)}(\lambda_n)}. \tag{5.9}$$

The last two relations and the fact that $\Phi_n(z)$ vanishes exactly $\mu_n - 1$ times at the point λ_n and μ_k times at the points λ_k, $k \neq n$, yield that (5.1) is valid. □

Lemma 5.2. *For each positive integer n, there exist bounded linear functionals V_n on the Banach space C_w and S_n^p on the Banach spaces L_w^p, with $p \in [1, \infty)$, such that*

$$\|V_n\| = \|S_n^p\| \leqslant E_n = \delta \exp\{-\lambda_n \sigma_\Lambda(\lambda_n) + A_6 \lambda_n\}, \tag{5.10}$$

where δ and A_6 are positive constants independent of n.

Proof. For each exponential polynomial $P(t) = \sum_k c_k t^{\mu_k - 1} e^{\lambda_k t}$, define

$$T_n(P) = \int_{-\infty}^{\infty} P(t) \Psi_n(t)\, dt,$$

where $\Psi_n(t)$ are the functions as in Lemma 5.1. It then follows that:

$$T_n(P) = c_n. \tag{5.11}$$

We will now prove that T_n is a bounded linear functional on the *span*(E_{Λ^*}) in $\| \|_{C_w}$ and $\| \|_{L^p_w}$. Let $\Phi_n(z)$ as in (5.3) and $G_\rho(z) = G(z+2\rho)$ where G is the function as in (4.14). Then, one gets that

$$\Phi_n^{(\mu_n-1)}(\lambda_n) = \frac{G_\rho^{(\mu_n)}(\lambda_n)}{\mu_n} \frac{f(\lambda_n)e^{-2A_4\lambda_n}(2\lambda_n+4\rho)e^{-2}}{(1+2\rho+\lambda_n)^{[6\rho D]+2}}.$$

One also deduces that $G_\rho^{(\mu_n)}(\lambda_n) = G^{(\mu_n)}(\lambda_n')$, with $\lambda_n' = \lambda_n + 2\rho$. Then from (1.28) and (4.13), one gets that

$$\begin{aligned}
|\Phi_n^{(\mu_n-1)}(\lambda_n)| &\geqslant \frac{|G^{(\mu_n)}(\lambda_n')|}{\mu_n!} \frac{|f(\lambda_n)|e^{-2A_4\lambda_n}(2\lambda_n+4\rho)e^{-2}}{(1+2\rho+\lambda_n)^{[6\rho D]+2}} \\
&\geqslant \exp\{\lambda_n'\sigma_{\Lambda'}(\lambda_n') - A_3\lambda_n'\} \frac{e^{-A_5\lambda_n}e^{-2A_4\lambda_n}(2\lambda_n+4\rho)e^{-2}}{(1+2\rho+\lambda_n)^{[6\rho D]+2}} \\
&\geqslant \exp\{\lambda_n\sigma_\Lambda(\lambda_n) - A_6\lambda_n\}
\end{aligned} \tag{5.12}$$

for some constant A_6. It follows from (5.9) and (5.12), that:

$$|\Psi_n(t)| \leqslant |H_n(t)| \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n\}. \tag{5.13}$$

Then for $t \geqslant 0$ one has from (5.6) that

$$|\Psi_n(t)e^{w(t)}| \leqslant \xi \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n - t\} \tag{5.14}$$

$$\leqslant \xi \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n\}, \tag{5.15}$$

and for $t < 0$ one has from (5.7) and (1.4) that

$$|\Psi_n(t)e^{w(t)}| \leqslant \xi' \exp\left\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n - \frac{\rho|t|}{2}\right\} \tag{5.16}$$

$$\leqslant \xi' \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n\}. \tag{5.17}$$

From (5.14) and (5.16) we get that there is some $\delta > 0$, independent of n, so that for $q \geqslant 1$ one has that

$$\left(\int_{-\infty}^{\infty} |\Psi_n(t)e^{w(t)}|^q \, dt\right)^{\frac{1}{q}} \leqslant \delta \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n\}. \tag{5.18}$$

Then, for $q = 1$ we get that

$$|T_n(P)| \leqslant \delta \|P\|_{C_w} \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n\}.$$

Also, from (5.15) (5.17) and (5.18), we get for all $p \in [1, \infty)$ that

$$|T_n(P)| \leqslant \delta \|P\|_{L^p_w} \exp\{-\lambda_n\sigma_\Lambda(\lambda_n) + A_6\lambda_n\}.$$

From the last two relations, one has that T_n is a bounded linear functional on the *span*(E_{Λ^*}) in the norms $\| \|_{C_w}$ and $\| \|_{L^p_w}$ for all $p \in [1, \infty)$.

Then by the Hahn–Banach theorem, T_n is extended to a bounded linear functional denoted by V_n on C_w and by S_n^p on L_w^p, such that

$$\|V_n\| = \|S_n^p\| = \|T_n\| \leqslant E_n = \delta \exp\{-\lambda_n \sigma_\Lambda(\lambda_n) + A_6 \lambda_n\}. \qquad \square$$

We are now ready to give the proof of Theorem 1.3.

5.1. *Part (A1), the closure of the linear span of E_{Λ^*} in C_w: case when* $\lim_{r\to\infty} \sigma_\Lambda(r) = \infty$

Suppose that a function f belongs to the closure of the *span*(E_{Λ^*}) in $\| \ \|_{C_w}$. Define

$$g(z) = \sum_{n=1}^{\infty} V_n(f) z^{\mu_n - 1} e^{\lambda_n z}, \tag{5.19}$$

where $\{V_n\}$ are the bounded functionals on C_w as defined in Lemma 5.2. We will prove that $f(x) = g(x)$ for all $x \in \mathbf{R}$ and that $g(z)$ is an entire function.

First we show that $g(z)$ is an entire function. We note that Valiron [20] proved that if for a multiplicity sequence $\{\lambda_n, \mu_n\}$ the following relations hold,

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0, \qquad \lim_{n\to\infty} \frac{\mu_n - 1}{\lambda_n} = 0, \tag{5.20}$$

then the regions of convergence of $p(z)$ where $p(z) = \sum_{n=1}^{\infty} V_n(f) e^{\lambda_n z}$, and $g(z)$ are identical, and both series converge absolutely for all z in the open region of convergence. In [23] (Lemma 3.5), we proved that for sequences $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$, relation (5.20) is valid for the (λ, μ) reordering of the sequence $\mathbf{B}$.

We claim that $p(z)$ is an entire function, thus the same holds for $g(z)$ as well, and as a result we have absolute convergence for both series for all $z \in \mathbf{C}$. Indeed, since $\sigma_\Lambda(r) \mapsto \infty$ as $r \mapsto \infty$ and (5.10) holds, then

$$|V_n(f)| \leqslant \|V_n\| \|f\|_{C_w} \leqslant E_n \|f\|_{C_w} = \delta \exp\{A_6 \lambda_n - \lambda_n \sigma_\Lambda(\lambda_n)\} \|f\|_{C_w}, \tag{5.21}$$

which implies that the $\limsup_{n\to\infty} \frac{\log |V_n(f)|}{\lambda_n} = -\infty$.

Second, we prove that $f(x) = g(x)$ for all $x \in \mathbf{R}$. Since $f \in \overline{span}(E_{\Lambda^*})$ in $\| \ \|_{C_w}$, there exists a sequence of exponential polynomials, call them $P_k(x)$, where $P_k(x) = \sum_{n=1}^{l(k)} a_{n,k} x^{\mu_n - 1} e^{\lambda_n x}$ with $l(k) \mapsto \infty$ as $k \mapsto \infty$, such that $\|f - P_k\|_{C_w} \mapsto 0$ as $k \mapsto \infty$. Fix now some $x \in \mathbf{R}$ and write

$$\begin{aligned} |f(x) - g(x)| &\leqslant |f(x) - P_k(x)| + |P_k(x) - g(x)| \\ &\leqslant e^{w(x)} \|f - P_k\|_{C_w} + \sum_{n=1}^{l(k)} |a_{n,k} - V_n(f)| |x|^{\mu_n - 1} e^{\lambda_n x} \\ &\quad + \sum_{n=l(k)+1}^{\infty} |V_n(f)| |x|^{\mu_n - 1} e^{\lambda_n x}. \end{aligned}$$

From Lemma 5.2 and with E_n as in (5.10) we have that

$$|V_n(f) - a_{n,k}| = |V_n(f) - V_n(P_k)| \leqslant E_n \|f - P_k\|_{C_w}.$$

Thus

$$|f(x)-g(x)| \leqslant e^{w(x)}\|f-P_k\|_{C_w} + \|f-P_k\|_{C_w}\sum_{n=1}^{l(k)} E_n|x|^{\mu_n-1}e^{\lambda_n x}$$
$$+\sum_{n=l(k)+1}^{\infty} |V_n(f)||x|^{\mu_n-1}e^{\lambda_n x}. \tag{5.22}$$

As in the case of $g(z)$ and $p(z)$, in a similar way we get that the region of absolute convergence of $U_1(z)=\sum_{n=1}^{\infty} E_n z^{\mu_n-1}e^{\lambda_n z}$ is the same with that of $U_2(z)=\sum_{n=1}^{\infty} E_n e^{\lambda_n z}$, which we deduce to be the whole complex plane. Thus, the sum $\sum_{n=1}^{l(k))} E_n|x|^{\mu_n-1}e^{\lambda_n x}$ has a finite limit as $k\mapsto\infty$. This fact, together with the relation $\|f-P_k\|_{C_w}\mapsto 0$ and the absolute convergence of $g(z)$ for all $z\in\mathbf{C}$, imply that letting $k\mapsto\infty$ in (5.22), yields that $f(x)=g(x)$. Since $x\in\mathbf{R}$ was arbitrary, then $f(x)=g(x)$ for all $x\in\mathbf{R}$.

5.2. Part (A2), the closure of the linear span of E_{Λ^*} in C_w: case when $\lim_{r\to\infty}\sigma_\Lambda(r)<\infty$

If $\sigma_\Lambda(r)$ is bounded, we get from (5.21) that there is some real constant M so that $\limsup_{r\to\infty}(\log|V_n(f)|)/\lambda_n\leqslant M$. Thus, this time we cannot deduce whether the $\limsup_{r\to\infty}(\log|V_n(f)|)/\lambda_n=-\infty$, and as a result we cannot claim that g as in (5.19) represents an entire function. We can only say that g is analytic in some half-plane instead. However, if w belongs to the class Ω as in (1.14), then g is indeed an entire function. In order to prove this, we need two auxiliary results, Lemmas 5.3 and 5.4.

Denote by L_k the *k-compositions* of logarithms, that is,

$$L_k(t)=\underbrace{\log\circ\log\circ\cdots\circ\log}_{k\text{-}times}(t). \tag{5.23}$$

Then denote by N_k the sequences

$$N_k=\left\{2n\prod_{j=1}^{k}L_j(n)\right\}_{n_k}^{\infty}, \tag{5.24}$$

that is, $N_1=\{2n\log n\}_{n_1}^{\infty}$, $N_2=\{2n\log n\log\log n\}_{n_2}^{\infty}$, etc., where the positive integers n_k are large enough so that every term of N_k is well defined. From the Euler–Maclaurin summation formula, the following result is valid.

Lemma 5.3. *For every positive integer k, the difference*

$$D_k(t)=\sum_{n_k<n\leqslant t}\frac{1}{n\prod_{j=1}^{k}L_j(n)}-L_{k+1}(t) \tag{5.25}$$

has a finite limit as $t\mapsto\infty$.

Lemma 5.4. *Let a function $w\in\Omega\cap\mathcal{A}_{\rho,\tau}$, with Ω as in* (1.14). *Let also for the sequence* $\mathbf{A}\in\mathbf{L}(\mathbf{c},\mathbf{0})$, *the sequence $\mathbf{B}\in\mathbf{A}_{\alpha,\beta}$ and $\Lambda=\{\lambda_n,\mu_n\}$ be its (λ,μ) reordering, with the* $\lim_{r\to\infty}\sigma_\Lambda(r)<\infty$. *Then there is a sequence U with real positive terms and $\Lambda\subset U$,* *so that the* $\lim_{r\to\infty}\sigma_U(r)=\infty$ *and*

$$\int_1^{+\infty}\frac{w(\sigma_U(t))}{1+t^2}\,dt<\infty. \tag{5.26}$$

Moreover, $U\in\mathbf{A}'_{\alpha,\beta}$ *for some sequence* $\mathbf{A}'\in\mathbf{L}(\mathbf{c},\mathbf{0})$.

Proof. Since $w\in\Omega$, by (1.14) there is a positive integer k so that $w(t)\leqslant E_k(t)$ with E_k as defined in (1.15). For this k consider the sequence N_k as in (5.24) and let $U=N_k\cup\Lambda$. We show now that (5.26) is valid. We get that

$$\sigma_{N_k}(t)=\sum_{2n\prod_{j=1}^k L_j(n)\leqslant t}\frac{2}{2n\prod_{j=1}^k L_j(n)}\leqslant\sum_{n\leqslant t}\frac{1}{n\prod_{j=1}^k L_j(n)}\leqslant L_{k+1}(t)+c_1,\quad c_1>0, \tag{5.27}$$

with the last inequality valid from Lemma 5.3. Since $U=N_k\cup\Lambda$ and $\sigma_\Lambda(t)$ is uniformly bounded, it follows that $\sigma_U(t)\leqslant L_{k+1}(t)+c_2$ for some $c_2>0$. By removing a finite number of terms from the sequence N_k, we can have $c_2=0$. Then we get that $w(\sigma_{U_k}(t))\leqslant w(L_{k+1}(t))\leqslant E_k(L_{k+1}(t))=\log t$, which implies that (5.26) is valid. We also note that the $\lim_{t\to\infty}\sigma_U(t)=\infty$ since the same holds for $\sigma_{N_k}(t)$.

Second, we show that $U\in\mathbf{A}'_{\alpha,\beta}$ for some sequence $\mathbf{A}'\in\mathbf{L}(\mathbf{c},\mathbf{0})$, where the class $\mathbf{L}(\mathbf{c},\mathbf{0})$ was defined in Section 1.1. By assumption, Λ is the (λ,μ) reordering of some multiplicity sequence $\mathbf{B}$, where $\mathbf{B}\in\mathbf{A}_{\alpha,\beta}$ (see Definition 1.3) and $\mathbf{A}\in\mathbf{L}(\mathbf{c},\mathbf{0})$. Then let $\mathbf{A}'=N_k\cup\mathbf{A}$, and write $\mathbf{A}'=\{a'_n\}_{n=1}^\infty$, with $a'_n<a'_{n+1}$. If required, we may subject the terms of the sequence N_k to bounded perturbations such that the terms a'_n satisfy $a'_{n+1}-a'_n>c'$ for some $c'>0$. Observe also that the sequence N_k has density zero, thus the same is true for $\mathbf{A}'$. Therefore $\mathbf{A}'\in\mathbf{L}(\mathbf{c}',\mathbf{0})$ and since $U=N_k\cup\mathbf{B}$, then $U\in\mathbf{A}'_{\alpha,\beta}$. □

Suppose now that f belongs to the closure of the $span(E_{\Lambda^*})$ in $\|\ \|_{C_w}$, with $w\in\Omega\cap\mathcal{A}_{\rho,\tau}$. Repeating the steps as in the proof of Theorem 1.3, part $(A1)$, we get that there is some function g which is analytic either in the whole complex plane $\mathbf{C}$ or in some half-plane $\mathcal{H}_{\mathcal{M}}=\{z:\Re z<M\}$, such that $f(x)=g(x)$ for all those real numbers x where g is defined at. We will assume that the second case holds, that is, g has a finite abscissa of convergence $\Re z=M$, and reach a contradiction.

Claim. *The abscissa is a natural boundary for* $g(z)$. *In other words, there is no open segment on the abscissa so that* $g(z)$ *can be extended analytically across.*

We now justify our claim. In [23] we proved that if the multiplicity sequence $\Lambda=\{\lambda_n,\mu_n\}_{n=1}^\infty$ has density zero, then the entire function $G(z)=\prod_{n=1}^\infty(1-\frac{z^2}{\lambda_n^2})^{\mu_n}$, is a function of exponential type zero and satisfies for every $\varepsilon>0$ as $n\mapsto\infty$ the estimate $\frac{\mu_n!}{|G^{(\mu_n)}(\lambda_n)|}=O(\exp\{\varepsilon\lambda_n\})$. This implies that $\mathbf{\Lambda}=\{\lambda_n,\mu_n\}_{n=1}^\infty$ is an interpolating variety for the space A_p^0 which is the space of entire functions of infraexponential type. Then from Berenstein and Gay [3, Chapter 6], as well as from [23, Theorem 2.4], we have that for Taylor–Dirichlet series with such frequencies, the abscissa of convergence is a natural boundary. Thus, our claim is justified.

Next, consider the sequence U as in Lemma 5.4, so that $\Lambda \subset U$. Then the function *f* belongs to the closure of the $span(E_{U^*})$ as well. From (5.26) and Theorem 1.1 we get that the system E_U is incomplete in C_w. Also from Lemma 5.4 we have that the $\lim_{r\to\infty} \sigma_U(r) = \infty$. Combining the two results, we get from Theorem 1.3, part $(A1)$, that $f(x)$ is extended to an entire analytic function $r(z)$ so that $f(x) = r(x)$ for all $x \in \mathbf{R}$.

Observe now that the functions $g(z)$ and $r(z)$ are both analytic in the half-plane $\mathcal{H}_{\mathcal{M}}$ and $g(x) = r(x)$ for all $x < M$. But this implies that $g(z) = r(z)$ for all $z \in \mathcal{H}_{\mathcal{M}}$. Thus, $r(z)$ is the analytic continuation of $g(z)$ from the half-plane $\mathcal{H}_{\mathcal{M}}$ to the entire complex plane. This contradicts the fact that the abscissa $\Re z = M$ is a natural boundary for $g(z)$. Therefore, $g(z)$ is an entire analytic function and admits the Taylor–Dirichlet series expansion (1.16).

5.3. *Part (B), the closure of the linear span of E_{Λ^*} in L^p_w*

Suppose now that a function *f* belongs to the closure of the $span(E_{\Lambda^*})$ in $\| \, \|_{L^p_w}$ for some $p \in [1, \infty)$. Define

$$h(z) = \sum_{n=1}^{\infty} S^p_n(f) z^{\mu_n - 1} e^{\lambda_n z}, \tag{5.28}$$

where $\{S^p_n\}$ are the bounded functionals on L^p_w as defined in Lemma 5.2. Then, $h(z)$ is an entire function. The proof is the same as that for $g(z)$ in (5.19). Our goal is to prove that $f(x) = h(x)$ for almost all $x \in \mathbf{R}$.

First recall relation (1.17), that is, $w(t) \geqslant e^{\delta t}$ for some $\delta > 0$, for all $t \geqslant 0$, and choose some positive integer n_0 so that $n_0 > \frac{1}{D\delta}$. Then, raise $w(x)$ to its n_0^{th} power, that is, $w^{n_0}(x)$ and denote this by $\Xi(x)$. Like $w(x)$, $\Xi(x)$ is also a convex function which belongs to the class $\mathcal{A}_{\rho',\tau'}$, and its Fenchel transform Ξ^* is well defined.

We will prove that $f(x) = h(x)$ for almost all $x \in \mathbf{R}$, by showing that

$$\left(\int_{-\infty}^{+\infty} |(f(x) - h(x)) e^{-\Xi(x)}|^p \, dx \right)^{\frac{1}{p}} = 0.$$

Since $f \in \overline{span}(E_{\Lambda^*})$ in $\| \, \|_{L^p_w}$, there exists a sequence of exponential polynomials $P_k(x) = \sum_{n=1}^{l(k)} a_{n,k} x^{\mu_n - 1} e^{\lambda_n x}$, with $l(k) \mapsto \infty$ as $k \mapsto \infty$ such that

$$\| f - P_k \|_{L^p_w} \mapsto 0, \quad k \mapsto \infty. \tag{5.29}$$

By applying the Minkwoski inequality one has that

$$\|(f - h) e^{-\Xi}\|_{L^p_w} \leqslant \|(f - P_k) e^{-\Xi}\|_{L^p_w} + \|(P_k - h) e^{-\Xi}\|_{L^p_w}. \tag{5.30}$$

We will show that the right-hand side of (5.30) tends to 0 as $k \mapsto \infty$, which implies that $f(x) = g(x)$ for almost all $x \in \mathbf{R}$. We note that $\|(f - P_k) e^{-\Xi}\|_{L^p_w}| \mapsto 0$ as $k \mapsto \infty$ since (5.29) holds. It remains to prove the same for $\|(P_k - h) e^{-\Xi}\|_{L^p_w}$. The following result is needed,

$$\sum_{n=1}^{\infty} E_n \left(\int_{-\infty}^{+\infty} |x|^{\mu_n - 1} e^{\lambda_n x} e^{-\Xi(x)} e^{-w(x)} |^p \, dx \right)^{\frac{1}{p}} < \infty, \tag{5.31}$$

where E_n is the constant as in (5.10). We will prove this relation by first deducing that

$$\left(\int_0^{+\infty}(x^{\mu_n-1}e^{\lambda_n x}e^{-\Xi(x)})^p\,dx\right)^{\frac{1}{p}} = \left(\int_0^{+\infty}(e^{(\lambda_n+1)x-\Xi(x)}x^{\mu_n-1}e^{-x})^p\,dx\right)^{\frac{1}{p}}$$
$$\leqslant e^{\Xi^*(\lambda_n+1)}\left(\int_0^{+\infty}(x^{\mu_n-1}e^{-x})^p\,dx\right)^{\frac{1}{p}}$$
$$\leqslant A_9 e^{\Xi^*(\lambda_{n+1})}[\mu_n p]! \tag{5.32}$$

for some $A_9>0$, with $[\mu_n p]$ the integral part of $\mu_n p$. Then, since $e^{\delta t}\leqslant w(t)$ one has that $e^{n_0\delta t}\leqslant\Xi(t)$, and this implies that $\Xi^*(t)\leqslant\frac{t}{\delta n_0}\log\frac{t}{\delta n_0}-\frac{t}{\delta n_0}$. Substitution in (5.32) yields for some $A_{10}\in\mathbf{R}$ that

$$\left(\int_0^{+\infty}(x^{\mu_n-1}e^{\lambda_n x}e^{-\Xi(x)})^p\,dx\right)^{\frac{1}{p}} \leqslant A_9[\mu_n p]!\,e^{\frac{\lambda_{n+1}}{\delta n_0}\log\frac{\lambda_{n+1}}{\delta n_0}-\frac{\lambda_{n+1}}{\delta n_0}}$$
$$\leqslant A_9[\mu_n p]!\,e^{D\lambda_n\log\lambda_n-A_{10}\lambda_n},$$

with the last inequality valid since $n_0D\delta>1$. By Lemma 1.1, one has that $\mu_n\leqslant\chi\lambda_n^{\alpha}$ for $\chi>0$ and $0<\alpha<1$. It follows that there is some $0<\alpha'<1$ so that $\mu_n p\leqslant\lambda_n^{\alpha'}$ for $n>n_0$. Thus $[\mu_n p]!\leqslant[\lambda_n^{\alpha'}]!<[\lambda_n^{\alpha'}]^{[\lambda_n^{\alpha'}]}$ for $n>n_0$ and

$$\left(\int_0^{+\infty}(x^{\mu_n-1}e^{\lambda_n x}e^{-\Xi(x)})^p\,dx\right)^{\frac{1}{p}} \leqslant A_9[\lambda_n^{\alpha'}]^{[\lambda_n^{\alpha'}]}e^{D\lambda_n\log\lambda_n-A_{10}\lambda_n}$$
$$= A_9e^{[\lambda_n^{\alpha'}]\log[\lambda_n^{\alpha'}]}e^{D\lambda_n\log\lambda_n-A_{10}\lambda_n}.$$

Since $E_n=\delta\exp\{-\lambda_n\sigma_\Lambda(\lambda_n)+A_6\lambda_n\}$ and from (1.9) one has for every $\varepsilon>0$ that $\sigma_\Lambda(\lambda_n)>(2D-\varepsilon)\log\lambda_n$ for all $n>n(\varepsilon)$, this implies that

$$\sum_{n=1}^{\infty}E_ne^{[\lambda_n^{\alpha'}]\log[\lambda_n^{\alpha'}]}e^{D\lambda_n\log\lambda_n-A_{10}\lambda_n}<\infty. \tag{5.33}$$

From Lemma 1.1 one has that $\mu_n<\lambda_n$, thus $(\int_{-\infty}^0(|x|^{\mu_n-1}e^{\lambda_n x})^p\,dx<1$. This fact and (5.33), imply convergence for the series in (5.31).

For convenience, we denote now $\|(P_k-h)e^{-\Xi}\|_{L_w^p}$ by I_k. Repeated application of the Minkwoski inequality gives that

$$I_k=\left\|\left(\sum_{n=1}^{l(k)}(a_{n,k}-S_n^p(f))x^{\mu_n-1}e^{\lambda_n x}-\sum_{n=l(k)+1}^{\infty}S_n^p(f)x^{\mu_n-1}e^{\lambda_n x}\right)e^{-\Xi(x)}\right\|_{L_w^p}$$
$$\leqslant\sum_{n=1}^{l(k)}\left\|\left((a_{n,k}-S_n^p(f))x^{\mu_n-1}e^{\lambda_n x}\right)e^{-\Xi(x)}\right\|_{L_w^p}$$
$$+\sum_{n=l(k)+1}^{\infty}\left\|\left(S_n^p(f)x^{\mu_n-1}e^{\lambda_n x}\right)e^{-\Xi(x)}\right\|_{L_w^p}. \tag{5.34}$$

If both sums on the right-hand side of (5.34) tend to 0 as $k\mapsto\infty$, then $I_k\mapsto 0$ as well, and we are done.

We consider the first sum, $\sum_{n=1}^{l(k)} \|((a_{n,k}-S_n^p(f))x^{\mu_n-1}e^{\lambda_n x}e^{-\Xi(x)}\|_{L^p_w}$ that we denote by $I_{k,1}$. With E_n as defined in (5.10), one gets that

$$|a_{n,k}-S_n^p(f)| = |S_n^p(P_k)-S_n^p(f)| \leqslant E_n\|f-P_k\|_{L^p_w}.$$

Thus

$$I_{k,1} \leqslant \|f-P_k\|_{L^p_w} \sum_{n=1}^{l(k)} E_n \left(\int_{-\infty}^{+\infty} |x^{\mu_n-1}e^{\lambda_n x}e^{-\Xi(x)}e^{-w(x)}|^p\,dx\right)^{\frac{1}{p}}.$$

Since the series in (5.31) converges and $\|f-P_k\|_{L^p_w} \mapsto 0$ as $k \mapsto \infty$, this implies that $I_{k,1} \mapsto 0$ as $k \mapsto \infty$.

We consider now the second sum that we denote by $I_{k,2}$. We get that

$$I_{k,2} \leqslant \sum_{n=l(k)+1}^{\infty} E_n\|f\|_{L^p_w} \left(\int_{-\infty}^{+\infty} |x^{\mu_n-1}e^{\lambda_n x}e^{-\Xi(x)}e^{-w(x)}|^p\,dx\right)^{\frac{1}{p}}.$$

Since the series in (5.31) converges, this implies that $I_{k,2} \mapsto 0$ as $k \mapsto \infty$. Thus, both sums, $I_{k,1}$ and $I_{k,2}$ tend to 0 as $k \mapsto \infty$, which implies the same for I_k.

6. Minimality of E_Λ in C_w and in L^p_w: proof of Theorem 1.4

In this section we prove that minimality of E_Λ is equivalent to incompleteness. The next lemma is the first step in order to deduce Theorem 1.4.

Lemma 6.1. *Let the sequence* $\mathbf{A} \in \mathbf{L}(\mathbf{c},\mathbf{D})$, *the sequence* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *and* $\Lambda = \{\lambda_n, \mu_n\}$ *be its* (λ,μ) *reordering. Let also* $w(t)$ *be a function which belongs to the class* $\mathcal{A}_{\rho,\tau}$. *Assume that the system* E_Λ *is not complete in* C_w *or in* L^p_w, *for some* $p \in [1,\infty)$. *Then, there exist analytic functions* $\{\Phi_{n,j}(z) : j = 0,1,2,\ldots,\mu_n-1\}_{n=1}^{\infty}$ *in the half-plane* $C_{-2\rho} = \{z : \Re z > -2\rho\}$, *so that*

$$\Phi_{n,j}^{(i)}(\lambda_n) = 0, \quad i \in \{0,1,2,\ldots,\mu_n-1\}\backslash\{\mu_n-j-1\}, \tag{6.1}$$

$$\Phi_{n,j}^{(\mu_n-j-1)}(\lambda_n) \neq 0, \tag{6.2}$$

and for all $k \neq n$ *one has that*

$$\Phi_{n,j}^{(i)}(\lambda_k) = 0, \quad i \in \{0,1,\ldots,\mu_k-1\}. \tag{6.3}$$

Furthermore, there are positive constants $M_{n,j}$ *so that* $\Phi_{n,j}(z)$ *satisfies the relation*

$$|\Phi_{n,j}(z)| \leqslant M_{n,j}\frac{\exp\{w^*(x)-A_4x\}}{(1+|z+2\rho|^2)}, \quad z \in C_{-2\rho}. \tag{6.4}$$

Proof. First, fix some positive integer n and some $j \in \{0,1,2,\ldots,\mu_n-1\}$. Let $G_\rho(z) = G(z+2\rho)$, where $G(z)$ is the meromorphic function as in (4.14), and define

$$G_{\rho,n,j}(z) = G_\rho(z)\left(\frac{z+\lambda_n+4\rho}{z-\lambda_n}\right)^{j+1} e^{-\frac{2(j+1)(z+2\rho)}{\lambda_n+2\rho}}.$$

Then $G_{\rho,n,j}(z)$ is analytic in the half-plane $C_{-2\rho}$ and vanishes exactly $\mu_n - j - 1$ times at the point λ_n, and μ_k times at the points λ_k, $k \neq n$. One can verify that $G_{\rho,n,j}(z)$ satisfies (1.26) for the same constant A_1 as $G_\rho(z)$ does.

Since E_Λ is incomplete, relation (1.10) holds. Then, from the proof in the sufficiency part of Theorem 1.1, there exists an analytic function $f(z)$ on $C_{-2\rho}$ with no zeros, such that (4.12) is satisfied. Define now

$$\phi_{n,j}(z) = \frac{G_{\rho,n,j}(z) f(z) e^{-2A_4 z}}{(1+2\rho+z)^{[6\rho D]+2}(\lambda_n + z + 4\rho)^{3\mu_n}}.$$

It follows that $\phi_{n,j}(z)$ is an analytic function in the half-plane $C_{-2\rho}$ and vanishes exactly $\mu_n - j - 1$ times at the point λ_n, and μ_k times at the points λ_k, $k \neq n$.

We claim now that there are unique constants $\{a_{l,n,j}\}$ so that if we define

$$\Phi_{n,j}(z) = \phi_{n,j}(z)\left[1 + \sum_{l=1}^{j-1} a_{l,n,j}(z-\lambda_n)^l\right], \tag{6.5}$$

then relations (6.1)–(6.4) are valid.

First observe, that for *any* constants $\{a_{l,n,j}\}$, $\Phi_{n,j}(z)$ vanishes exactly $\mu_n - j - 1$ times at the point λ_n, and at *least* μ_k times at the point λ_k, $k \neq n$. For this reason, (6.2)–(6.3) are valid and (6.1) holds for $i \in \{0, 1, \ldots, \mu_n - j - 2\}$. It remains to prove that there exist unique constants $\{a_{l,n,j}\}$, so that (6.1) holds for $i \in \{\mu_n - j, \mu_n - j + 1, \ldots, \mu_n - 1\}$ as well.

For $i \in \{\mu_n - j, \mu_n - j + 1, \ldots, \mu_n - 1\}$ one deduces that

$$\Phi^{(i)}_{n,j}(\lambda_n) = \sum_{l=0}^{i-\mu_n+j+1} \frac{i!}{(i-l)!}\phi^{(i-l)}_{n,j}(\lambda_n) a_{l,n,j},$$

since $\phi^{(i)}_{n,j}(\lambda_n) = 0$ for all $i \in \{0, 1, \ldots, \mu_n - j - 2\}$. Thus we have that

$$\Phi^{(\mu_n-j)}_{n,j}(\lambda_n) = \sum_{l=0}^{1} \frac{(\mu_n - j)!}{(\mu_n - j - l)!}\phi^{(\mu_n-j-l)}_{n,j}(\lambda_n) a_{l,n,j},$$

$$\Phi^{(\mu_n-j+1)}_{n,j}(\lambda_n) = \sum_{l=0}^{2} \frac{(\mu_n - j + 1)!}{(\mu_n - j + 1 - l)!}\phi^{(\mu_n-j+1-l)}_{n,j}(\lambda_n) a_{l,n,j},$$

and so on. We then determine the unique constants $a_{l,n,j}$ so that $\Phi^{(i)}_{n,j}(\lambda_n) = 0$ for $i \in \{\mu_n - j, \mu_n - j + 1, \ldots, \mu_n - 1\}$. We get that

$$a_{1,n,j} = \frac{-\phi^{(\mu_n-j)}_{n,j}(\lambda_n)}{(\mu_n - j)\phi^{(\mu_n-j-1)}_{n,j}(\lambda_n)},$$

$$a_{2,n,j} = \frac{-\phi^{(\mu_n-j+1)}_{n,j}(\lambda_n) - (\mu_n - j + 1)\phi^{(\mu_n-j)}_{n,j}(\lambda_n) a_{1,n,j}}{(\mu_n - j + 1)(\mu_n - j)\phi^{(\mu_n-j-1)}_{n,j}(\lambda_n)},$$

etc. Observe that since $\phi^{(\mu_n-j-1)}_{n,j}(\lambda_n) \neq 0$ the constants $\{a_{l,n,j}\}$ are well defined, thus (6.1) is valid for all $i \in \{0, \ldots, \mu_n - 1\}$.

Now, let $M_{n,j} = \max\{|a_{l,n,j}| : l = 1, 2, \ldots, j-1\}$ and define

$$R_{n,j}(z) = \frac{1 + \sum_{l=1}^{j-1} a_{l,n,j}(z-\lambda_n)^l}{(\lambda_n + z + 4\rho)^{3\mu_n}}.$$

Observe that for all $z \in C_{-2\rho}$ one has that $|\lambda_n + z + 4\rho| \geqslant \lambda_n$ and $|\lambda_n - z|/|\lambda_n + z + 4\rho| \leqslant 1$. Thus, $|R_{n,j}(z)| \leqslant M_{n,j}$ for all $z \in C_{-2\rho}$. Combining this result with (5.4) gives (6.4). □

Lemma 6.2. *There exist continuous functions $\{\Psi_{n,j}(t)\colon\ j = 0, 1, 2, \ldots, \mu_n - 1\}_{n=1}^{\infty}$ on the real line $\mathbf{R}$, so that*

$$\int_{-\infty}^{\infty} \Psi_{n,j}(t) t^i e^{\lambda_k t}\, dt = \begin{cases} 1, & k = n,\ \ i = \mu_n - j - 1, \\ 0, & k = n,\ \ i \in \{0, 1, \ldots, \mu_n - 1\} \backslash \{\mu_n - j - 1\}, \\ 0, & k \neq n,\ \ i \in \{0, 1, \ldots, \mu_k - 1\}. \end{cases} \tag{6.6}$$

Proof. From (6.4), we deduce by contour integration that the value of the integral

$$H_{n,j}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi_{n,j}(x+iy) e^{-(x+iy)t}\, dy, \quad x > -2\rho,$$

does not depend on x, thus it is a function of t only. This function is continuous on $\mathbf{R}$, and since the upper bound in (6.4) differs by a factor of $M_{n,j}$ compared to the one in (4.15), then

$$|H_{n,j}(t)| \leqslant M_{n,j} \xi \exp\{-w(t) - t\} \quad \forall t \geqslant 0, \tag{6.7}$$

and

$$|H_{n,j}(t)| \leqslant M_{n,j} \xi' e^{-\frac{3}{2}\rho|t|} \quad \forall t < 0. \tag{6.8}$$

These relations imply that for every fixed $x > -\rho$, $e^{xt} H_{n,j}(t) \in L^1(-\infty, \infty)$, thus $\Phi_{n,j}(z)$ is equal to the Inverse Fourier Transform of $H_{n,j}(t)e^{tx}$, that is,

$$\Phi_{n,j}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_{n,j}(t) e^{tz}\, dt, \quad x > 0.$$

Next, define

$$\Psi_{n,j}(t) = \frac{H_{n,j}(t)}{\Phi_{n,j}^{(\mu_n - j - 1)}(\lambda_n)}. \tag{6.9}$$

Then from Lemma 6.1 and (6.9), we get that (6.6) is valid. □

Lemma 6.3. *For each positive integer n and each integer $j \in \{0, 1, 2, \ldots, \mu_n - 1\}$, there exist non-trivial bounded linear functionals $V_{n,j}$ on C_w and $S_{n,j}^p$ on L_w^p, for $p \in [1, \infty)$, so that*

$$V_{n,j}(t^i e^{\lambda_k t}) = \begin{cases} 1, & k = n,\ \ i = j, \\ 0, & k = n,\ \ i \in \{0, 1, \ldots, \mu_n - 1\} \backslash \{j\}, \\ 0, & k \neq n,\ \ i \in \{0, 1, \ldots, \mu_k - 1\}. \end{cases} \tag{6.10}$$

Similarly for $S_{n,j}^p$.

Proof. For each exponential polynomial $P(t) = \sum_k (\sum_{i=0}^{\mu_k - 1} c_{k,i} t^i) e^{\lambda_k t}$, define

$$T_{n,j}(P) = \int_{-\infty}^{+\infty} P(t) \Psi_{n,\mu_n - j - 1}(t)\, dt = c_{n,j},$$

where $\Psi_{n,j}(t)$ are the functions in the previous lemma. Relations (6.7)–(6.9), yield that $T_{n,j}$ is a bounded linear functional on the $span(E_\Lambda)$ in $\| \ \|_{C_w}$ and in $\| \ \|_{L^p_w}$ for all $p \in [1, \infty)$. Then, by the Hahn–Banach theorem, $T_{n,j}$ is extended to a bounded linear functional denoted by $V_{n,j}$ on C_w and by $S^p_{n,j}$ on L^p_w. Finally, (6.10) follows from (6.6). $\square$

Proof of Theorem 1.4. First assume that E_Λ is minimal in C_w. Then for any $t^i e^{\lambda_n t} \in E_\Lambda$, the system $E_\Lambda \backslash \{t^i e^{\lambda_n t}\}$ is incomplete in C_w. By the necessary and sufficient conditions of Theorem 1.1, it follows that E_Λ is also incomplete in C_w. Similarly for L^p_w.

Next, assume that E_Λ is incomplete but not minimal in C_w. Then, there is a positive integer n and some integer $j \in \{0, 1, 2, \ldots, \mu_n - 1\}$ so that $t^j e^{\lambda_n t} \in \overline{span}(E_\Lambda \backslash \{t^j e^{\lambda_n t}\})$ in $\| \ \|_{C_w}$. In other words, for every $\varepsilon > 0$ there is an exponential polynomial $\sum_k \sum_i a_{k,i} t^i e^{\lambda_k t}$, call it $P_{k,i}(t)$, where each $t^i e^{\lambda_k t} \neq t^j e^{\lambda_n t}$, so that $\|(t^j e^{\lambda_n t} - P_{k,i}(t))\|_{C_w} < \varepsilon$. Let $V_{n,j}$ be the bounded functional of Lemma 6.3. From (6.10) one has that $V_{n,j}(t^j e^{\lambda_n t} - P_{k,i}(t)) = 1$, and since $V_{n,j}$ is bounded, $|V_{n,j}(t^j e^{\lambda_n t} - P_{k,i}(t))| \leqslant \|t^j e^{\lambda_n t} - P_{k,i}(t)\|_{C_w} \|V_{n,j}\| \leqslant \varepsilon \|V_{n,j}\|$. Thus, $1 \leqslant \varepsilon \|V_{n,j}\|$ for every $\varepsilon > 0$, a contradiction. Similarly for L^p_w. $\square$

Acknowledgments

The author expresses his deep appreciation to Dr. A. Vidras for his useful suggestions and remarks, as well as to the referees for their valuable comments.

References

[1] J.M. Anderson, K.G. Binmore, Closure theorems with applications to entire functions with gaps, Trans. Amer. Math. Soc. 161 (1971) 381–400.
[2] R.C. Baker, G. Harman, J. Pintz, The difference between consecutive primes, II, Proc. London Math. Soc. 83 (3) (2001) 532–562.
[3] C.A. Berenstein, R. Gay, Complex Analysis and Special Topics in Harmonic Analysis, Springer, New York, 1996 x+482pp. ISBN:0-387-94411-7..
[4] R.P. Boas Jr., Entire Functions, Academic Press, New York, 1954.
[5] A. Borichev, On weighted polynomial approximation with monotone weights, Proc. Amer. Math. Soc. 128 (12) (2000) 3613–3619.
[6] A. Borichev, On the closure of polynomials in weighted spaces of functions on the real line, Indiana Univ. Math. J. 50 (2) (2001) 829–846.
[7] P. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New-York, 1995 x+480pp. ISBN: 0-387-94509-1.
[8] L. De Branges, The Bernstein problem, Proc. Amer. Math. Soc. 10 (1959) 825–832.
[9] L. Carleson, On Bernsteins approximation problem, Proc. Amer. Math. Soc. 2 (1951) 953–962.
[10] G.T. Deng, Une condition necessaire et suffisante pour la quasi-analyticite de Mandelbrojt sur une demi-droite, C. R. Acad. Sci. Paris Ser. I Math. 306 (19) (1988) 769–772.
[11] G.T. Deng, Incompleteness and closure of a linear span of exponential system in a weighted Banach space, J. Approx. Theory 125 (1) (2003) 1–9.
[12] W.H.J. Fuchs, On the closure of $\{e^{-t} t^{a_\nu}\}$, Proc. Cambridge Philos. Soc. 42 (1946) 91–105.
[13] W.H.J. Fuchs, An application of the Ahflors distortion theorem, J. Anal. Math. 18 (1967) 61–79.
[14] P. Koosis, The Logarithmic Integral, vol. I, Cambridge University Press, Cambridge, 1988.
[15] P. Malliavin, Sur quelques procédés d'extrapolation, Acta Math. 93 (1955) 179–255.

[16] H. Pollard, The Bernstein approximation problem, Proc. Amer. Math. Soc. 6 (1955) 402–411.
[17] R. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[18] A.M. Sedletskii, Analytic Fourier transforms and exponential approximations I, J. Math. Sci. (New York) 129 (6) (2005) 4251–4408.
[19] A.M. Sedletskii, Analytic Fourier transforms and exponential approximations II, J. Math. Sci. (New York) 130 (6) (2005) 5083–5254.
[20] M.G. Valiron, Sur les solutions des équations différentielles linéaires d'ordre infini et a coefficients constants, Ann. Ecole Norm. 46 (3) (1929) 25–53.
[21] B.V. Vinnitskii, A.V. Shapovalovskii, Completeness of exponentials with weight, Ukrainian Math. J. 41 (12) (1989) 1464–1469.
[22] B.V. Vinnitskii, A.V. Shapovalovskii, A remark on the completeness of exponentials with weight in $L^2(\mathbf{R})$, Ukrainian Math. J. 52 (7) (2000) 1002–1009.
[23] E. Zikkos, On a theorem of Norman Levinson and a variation of the Fabry Gap theorem, Complex Variables 50 (4) (2005) 229–255.